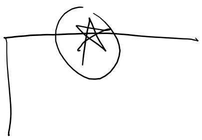
DG2020/02/10

Monday, February 10, 2020 11:34 AM

Previously, we used averages for parabolic heat conduction. Here, we are solving an elliptic PDE (CT dot term is absent from the PDE). So, we need to use elliptic fluxes.
In Arnold 2000, Arnold 20002 papers, many different formulas are given for star values of elliptic PDEs.
A general form can be given as follows:



$$\boxed{\begin{aligned} q^* &= \{\{q\}\} + \vec{\beta}\,[[q]] + \underbrace{K\alpha}_{\bar{\alpha}}[[T]] \\ T^* &= \{\{T\}\} + \vec{\gamma}\cdot[[T]] \end{aligned}}$$

LDG fluxes
"Cockburn & Shu"

$\sigma = \nabla u$

Method	$h_\sigma^{e,K}$	$h_u^{e,K}$
Bassi–Rebay 1	$\{\sigma_h\}$	$\{u_h\}$
Brezzi et al. 1	$\{\sigma_h\} - \eta^e\{r_e([\![u_h]\!])\}$	$\{u_h\}$
LDG	$\{\sigma_h\} - \eta^e[\![u_h]\!] + \beta^e[\![\sigma_h]\!]$	$\{u_h\} + \gamma^e[\![u_h]\!]$

We'll get back to (*) to discuss the effect of α, β, γ

$-\ \bar{\alpha} = K\alpha$

$[\{\{q\}\}] = [K\alpha[[T]]]$ Dimension

Dimension of $\alpha \leftarrow \dfrac{1}{L}$

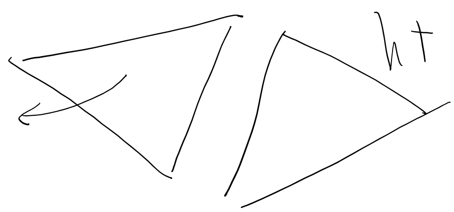
$q = -K\nabla T$

$\dfrac{[T]}{[L]}$

$$\boxed{\alpha = \frac{\eta_0}{h_{min}}}$$

$h_{min} = \min\{h^-, h^+\}$

length scale



h^- h^+

$$\boxed{\alpha \propto \frac{1}{h}}$$

for LDG (*)

$\eta_0 > 0$ is sufficient for stability

Arnold 2002 paper

TABLE 6.1
Properties of the DG methods

Method	cons.	a.c.	stab.	type	cond.	H^1	L^2
Brezzi et al. [18]	✓	✓	✓	α^r	$\eta_0 > 0$	h^p	h^{p+1}
LDG [35]	✓	✓	✓	α^j	$\eta_0 > 0$	h^p	h^{p+1}
IP [43]	✓	✓	✓	α^j	$\eta_0 > \eta^*$	h^p	h^{p+1}
Bassi et al. [10]	✓	✓	✓	α^r	$\eta_0 > 3$	h^p	h^{p+1}

IP [43]	✓	✓	✓	α^j	$\eta_0 > \eta^*$	h^p	h^{p+1}
Bassi et al. [10]	✓	✓	✓	α^r	$\eta_0 > 3$	h^p	h^{p+1}
NIPG [53]	✓	×	✓	α^j	$\cancel{\eta_0 > 0}$	h^p	h^p
Babuška–Zlámal [6]	×	×	✓	α^j	$\eta_0 \approx h^{-2p}$	h^p	h^{p+1}
Brezzi et al. [19]	×	×	✓	α^r	$\eta_0 \approx h^{-2p}$	h^p	h^{p+1}
Baumann–Oden ($p = 1$)	✓	×	×	-	-	×	×
Baumann–Oden ($p \geq 2$)	✓	×	×	-	-	h^p	h^p
Bassi–Rebay [9]	✓	✓	×	-	-	$[h^p]$	$[h^{p+1}]$

horrible

$\vec{\gamma}, \vec{\beta}$ are two vectors, generally $\vec{\gamma} = -\vec{\beta}$

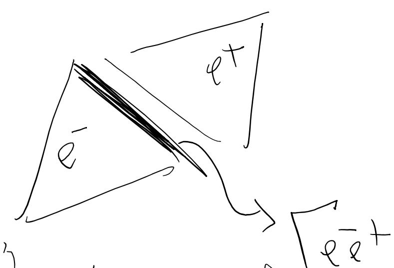
We'll talk about how these two vectors can result in an alternating flux scheme.

For the moment, assume $\vec{\gamma}, \vec{\beta} = 0$

$$\boxed{\begin{aligned} q^* &= \{\{q\}\} + \bar{\alpha}[[T]] \\ T^* &= \{\{T\}\} \end{aligned}}$$

$(*')$ LDG for $\vec{\beta} = \vec{\gamma} = 0$

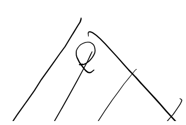
Recall
$$R_{\Gamma_{ee^+}}(\hat{T}, T) = B_{\Gamma_{ee^-}}(\hat{T}, T) = \varepsilon \int_{\Gamma_{\bar{e}e^+}} [[\hat{q}]](T^* - \{\{T\}\}) - \{\{\hat{q}\}\}[[T]]\, ds + \int [[\hat{T}]]\, q^*\, ds.$$



plug $(*')$ into $B(\hat{T}, T)$

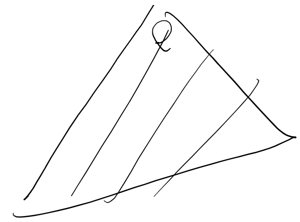
$$\boxed{B_{\Gamma_{\bar{e}e^+}}(\hat{T}, T) = \int_{\Gamma_{\bar{e}e^+}} [[\hat{T}]]\left(\{\{q\}\} + \bar{\alpha}[[T]]\right) ds + \varepsilon \int_{\Gamma_{\bar{e}e^+}} \left(-\{\{\hat{q}\}\}[[T]]\right)} \quad (1)$$

Contribution from inside the element



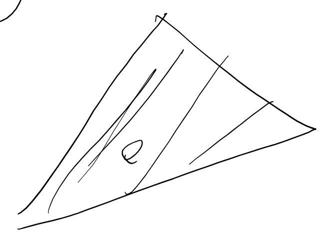
$B(\hat{T}, T) = \int (\varepsilon \nabla \hat{T} q - \hat{T} Q) \, dV$

$$R(\hat{T}, T) = \int_e \left(-\nabla \hat{T} \cdot q - \hat{T} Q\right) dV$$

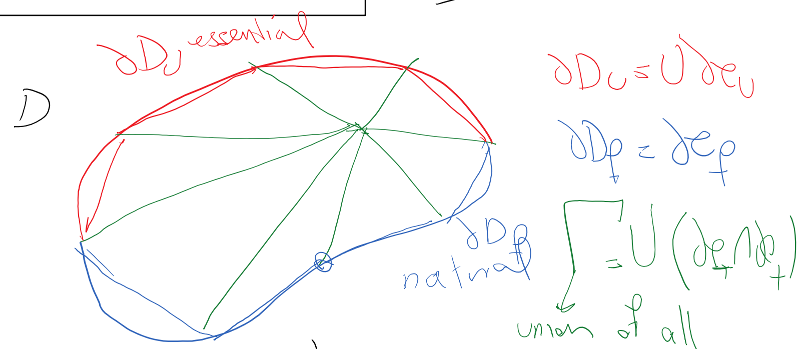


$$= \underbrace{\int_e \nabla \hat{T}\, k\, \nabla T \, dV}_{\text{goes to bilinear form}} \; - \underbrace{\int_e \hat{T} Q \, dV}_{\text{goes to RHS}}$$

$$\boxed{\begin{aligned} B_i^e(\hat{T}, T) &= \int_e \nabla \hat{T}\, k\, \nabla T \, dV \\ L_i^e(\hat{T}) &= \int_e \hat{T} Q \, dV \end{aligned} \quad \text{inside the element}} \quad (2)$$



Now we need to add up all the contributions from
Essential BC (last time)
Natural BC (last time)
Interior interfaces (finished this time (equation 1))
Interior of the elements (this time, equation 2))
To get the global system of equations



$D^h = \cup e$ the interior of all element

Find T such that for all $\hat{T}$:

$$B(\hat{T},T) = L(\hat{T})$$

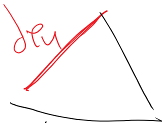
$$B(\hat{T},T) = \sum_e \underbrace{B_i^e(\hat{T},T)}_{\text{interior of elements}} + \sum_{\partial e_u} B_u^e(\hat{T},T) + \sum_{\Gamma_{ee^+}\in\Gamma} \underbrace{B_{\Gamma_{ee^+}}(\hat{T},T)}_{\text{interior interfaces}}$$

$$L(\hat{T}) = \sum_e L_i^e(\hat{T}) + \sum_{\partial e_u} L_u^e(\hat{T}) + \sum_{\partial e_f} L_f^e(\hat{T})$$

natural BC

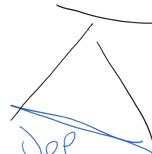
$$B_i^e = \int -\nabla\hat{T}\,q = \int \nabla\hat{T}\,k\,\nabla T\,dv \qquad L_i^e = \int \hat{T}Q\,dv \qquad \text{interior of element}$$

$$B_u^e = \int_{\partial e_u} (\hat{T}q - \varepsilon T\hat{q})\,ds, \qquad L_u^e(\hat{T}) = -\varepsilon \int_{\partial e_u} \hat{q}\cdot n\,\bar{T}\,ds$$



essential BC

$$\cancel{B_f^e(\hat{T},T)} = 0 \qquad L_f^e(\hat{T}) = -\int_{\partial e_f} \hat{T}\,\bar{q}\,ds$$

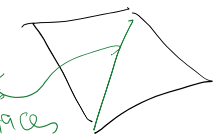


natural BC

$$B_\Gamma^{e\bar{e}^+} = \int_{\Gamma_{ee^+}} [[\hat{T}]]\,q^*\,ds + \varepsilon \int [\hat{q}]\left(T^* - \{T\}\right) - \{\hat{q}\}[T]\,ds$$

for the choice $T^* = \{T\}$, $q^* = \{q\} + \bar{\alpha}[T]$ we get

$$B_\Gamma^{e\bar{e}^+} = \int_{\Gamma_{e\bar{e}^+}} [\hat{T}]\left(\{q\} + \bar{\alpha}[T]\right)ds + \varepsilon \int -\{\hat{q}\}[T]\,ds$$



Bilinear statement (BLS)

Arnold 2002

Method	$B_h(u,v)$
Bassi–Rebay [9]	$(\nabla_h u + R(u), \nabla_h v + R(v))$
Brezzi et al. [18]	$(\nabla_h u + R(u), \nabla_h v + R(v)) + \alpha^r(u,v)$
LDG [35]	$(\nabla_h u + R(u) + L_\beta(u), \nabla_h v + R(v) + L_\beta(v)) + \alpha^j(u,v)$
IP [43]	$(\nabla_h u, \nabla_h v) + (R(u), \nabla_h v) + (\nabla_h u, R(v)) + \alpha^j(u,v)$
Bassi et al. [10]	$(\nabla_h u, \nabla_h v) + (R(u), \nabla_h v) + (\nabla_h u, R(v)) + \alpha^r(u,v)$
Baumann–Oden [12]	$(\nabla_h u, \nabla_h v) - (R(u), \nabla_h v) + (\nabla_h u, R(v))$
NIPG [53]	$(\nabla_h u, \nabla_h v) - (R(u), \nabla_h v) + (\nabla_h u, R(v)) + \alpha^j(u,v)$
Babuška–Zlámal [6]	$(\nabla_h u, \nabla_h v) + \alpha^j(u,v)$
Brezzi et al. [19]	$(\nabla_h u, \nabla_h v) + \alpha^r(u,v)$

we've explicitly added BC contributions

3 discussion points on bilinear form:

1. Symmetry of stiffness K
2. Is the bilinear form coercive (B(T,T) >= c |T|)? -> Is related to the stability of method
3. Explicitly expand these equations for 1D case so we can more clearly see how K and F look like in discrete form. This will be used in your HW assignments.

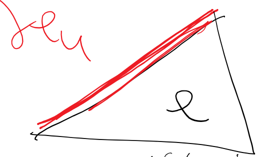
Symmetry:

Symmetry:

Look at stiffness from all 4 contributions:

a. Essential BC

$$B_u(\hat{T},T) = \int_{\partial e_u} (\hat{T} q - \varepsilon T \hat{q}) \cdot n \, ds$$



$$T = [\underbrace{T_1(x)\ T_2(x) \dots T_n(x)}_{\text{basis functions}}] \begin{bmatrix} a_1(t) \\ a_2(t) \\ \vdots \\ a_n(t) \end{bmatrix} \longrightarrow q = -k\nabla T = [q_1(x) \dots q_n(x)] \begin{bmatrix} a_1(t) \\ \vdots \\ a_n(t) \end{bmatrix}$$

$a_i(t)$: parabolic PDE $\quad a_i$: Elliptic PDE

$$\boxed{q_i = -k\nabla T_i}$$

basis functions for q columns are sln

K_{ij} = ? from $B_u(\hat{T},T)$

$$\boxed{K_{ij} = B_u(T_i, T_j)}$$

$\hat{T} = T_i \quad T = T_j$

$$\begin{array}{c} \\ \text{rows weights} \\ \text{row } i \ \hat{T}_i \end{array} \begin{array}{c} T_j \text{ column } j \\ \left[\quad \quad \quad \right] \end{array}$$

$$K_{ij} = \int (T_i q_j - \varepsilon T_j q_i) \cdot n \, ds$$

$$K_{ji} = \int (T_j q_i \boxed{-\varepsilon T_i q_j}) \, n \, ds \qquad i \rightleftarrows j$$

$\varepsilon = -1$	$K_{ji} = K_{ij}$	sym
$\varepsilon = 1$	$K_{ji} = -K_{ij}$	skew sym
$\varepsilon = 0$ & any other ε	neither one	

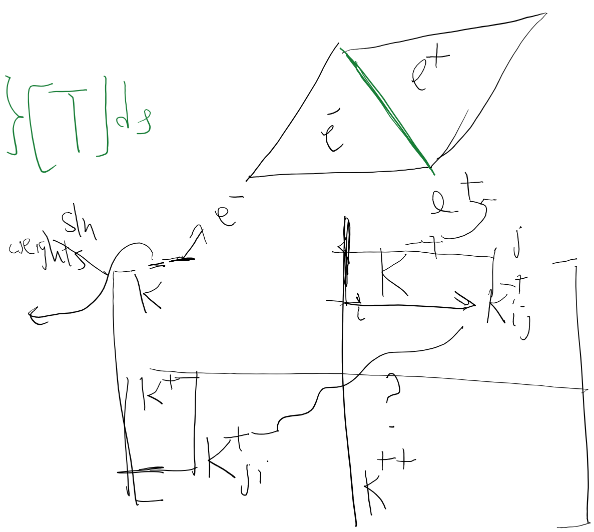
Essential BC

① sym or not

$\varepsilon = -1$ is what we did in class in the beginning of the course

sym or not interior interfaces:

$$\underbrace{B_{e\bar{e}^+}}_{B(\hat{T},T)} = \int_{\Gamma_{e^-e^+}} [\hat{T}] (\{q\} + \bar{\alpha}[T]) \, ds + \varepsilon \int_{\Gamma_{e^-e^+}} \{\hat{q}\}[T] \, ds$$



$$K_{ij}^{-+} = B(\hat{T},T)$$

for $\hat{T} = T_i^-$

$T = T_j^+$

let's compute this $\quad K_{ij}^{-+} \quad \hat{T} = T_i^- \quad T = T_j$

$$[\hat{T}] = \hat{T}^- n^- + \hat{T}^+ n^+ = T_i n^- + 0 \, n^+ = T_i n^-$$

$$[T] = T^- n^- + T^+ n^+ = 0 \, n^- + T_j n^+ = T_j n^+$$

$$\{\hat q\} = \frac{\hat q^- + \hat q^+}{2} = \frac{q_i^-}{2} + 0 = q_i^-/2$$

$$\{q\} = \frac{q^- + q^+}{2} = 0 + \frac{q_j^+}{2} = q_j^+/2$$

plug these 4 in bilinear form to get:

$$\underbrace{B_{e^-e^+}}_{B(\bar T, T)} = \int_{\Gamma_{e^-e^+}} [\hat T](\{q\} + \bar\alpha [T])\,ds + \varepsilon \int_{\Gamma_{e^-e^+}} -\{\hat q\}[T]\,ds$$

$$K_{ij}^{-+} = \int_{\Gamma_{e^-e^+}} (T_i^- n^-)\left(\frac{q_j^+}{2} + \bar\alpha T_j^+ n^+\right) ds + \varepsilon \int_{\Gamma_{e^-e^+}} -\frac{q_i^-}{2} T_j^+ n^+$$

$$= \int_{\Gamma_{e^-e^+}} \left[\bar\alpha\, T_i^- T_j^+ \underbrace{n^- \cdot n^+}_{-1} + \frac{1}{2}\left(T_i^- q_j^+ n^- - \varepsilon q_i^- T_j^+ \underbrace{n^+}_{-n^-}\right)\right] ds$$

$$K_{ij}^{-+} = \int_{\Gamma_{e^-e^+}} \Big[\underbrace{\boxed{-\bar\alpha\, T_i^- T_j^+}}_{a} + \frac{n^-}{2}\Big(\underbrace{\boxed{T_i^- q_j^+}}_{b} + \boxed{\varepsilon\, T_j^+ q_i^-} \Big)\Big] ds$$

$$K_{ji}^{+-} = \int_{\Gamma_{e^-e^+}} -\bar\alpha\, T_j^+ T_i^- + \frac{n^+}{2}\left(T_j^+ q_i^- + \varepsilon\, T_i^- q_j^+\right) ds$$

$-n^-$

$i \rightleftarrows j$

$- \longrightarrow +$

$+ \longrightarrow -$

$$= \int \underbrace{\boxed{-\bar\alpha\, T_i^- T_j^+}}_{a} + \frac{n^-}{2}\Big(\underbrace{\boxed{-\varepsilon\, T_i^- q_j^+}}_{b} \underbrace{\boxed{-T_j^+ q_i^-}}_{c} \Big) ds$$

a part $\longrightarrow$ equal (sym)

b, c

$\varepsilon = -1$ sym

$\varepsilon = 1$ skew sym

ε other neither one

$\alpha = 0$ $\varepsilon = 1$ skew sym

$\varepsilon = -1$ sym

else neither one