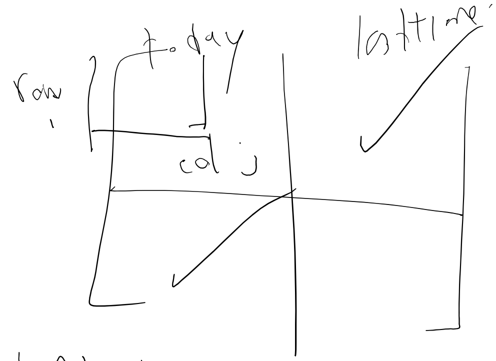
DG2020/02/12

Wednesday, February 12, 2020 11:39 AM

How about (K--)_ij for interior interfaces

$$B_{\Gamma_{e^-e^+}} = \int_{\Gamma_{e^-e^+}} [\hat{T}]\left(\{\hat{q}\} + \bar{\alpha}[T]\right) ds + \varepsilon \int_{\Gamma_{e^-e^+}} \{\hat{q}\}[T]\, ds$$

weigh sln



$$K^{--}_{ij} = B_{\Gamma_{e^-e^+}}(\hat{T}_i, T_j), \quad [\hat{T}] = T_i^- n^-, \; \{\hat{q}\} = \tfrac{1}{2} q_i^-, \; [T] = T_j^- n^-, \; \{q\} = \tfrac{1}{2} q_j^-$$

$$K^{--}_{ij} = \int_{\Gamma_{e^-e^+}} T_i^- n^-\left(\tfrac{1}{2} q_j^- + \bar{\alpha} T_j^- n^-\right) + \varepsilon\left(q_i^-\right)\left(T_j^- n^-\right) ds$$

$$K^{--}_{ij} = \int_{\Gamma_{e^-e^+}} \bar{\alpha} T_i^- T_j^- + n^-\left(T_i^- q_j^- + \varepsilon T_j^- q_i^-\right) ds$$

$$\downarrow \downarrow$$

$$j \quad i$$

$$K^{--}_{ji} = \int_{\Gamma_{e^-e^+}} \bar{\alpha} T_j^- T_i^- + n^-\left(\varepsilon T_i^- q_j^- + T_j^- q_i^-\right) ds$$

for $\varepsilon = -1$ fully symmetric

for $\varepsilon = 1$ & $\alpha = 0$ skew sym

else neither one

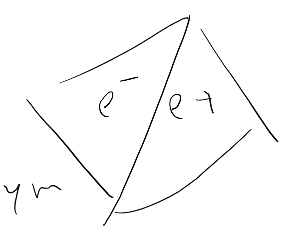
(2)

for interfaces

$\varepsilon = -1$ sym

$\varepsilon = 1$ & $\alpha = 0$ skew sym

else neither



$$\begin{bmatrix} K^{--} & K^{-+} \\ K^{+-} & K^{++} \end{bmatrix}$$

3) Natural: no contribution to $B_{(3)}$

4) Interior of the element:

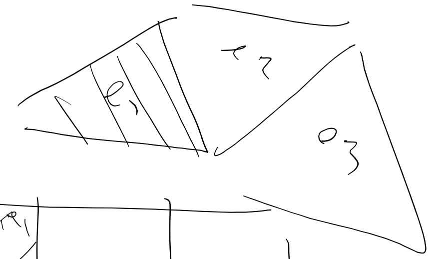
4) Interior of the element:

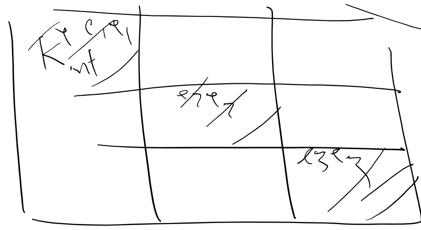
$$B^e_{int}(\hat{T},T) = \int \nabla \hat{T} \, \cancel{k} \, \nabla T \, dv$$

$$K^{ee}_{\cancel{int}\,ij} = \int \nabla T_i \, k \, \nabla T_j \, dv$$

sym matrix

(4) sym.





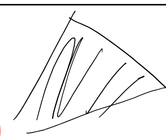
From (1) to (4)

$$\begin{cases} \varepsilon = -1 & \text{symmetric} \\ \varepsilon = 1 \ \& \ \alpha = 0 & \text{except interior contribution skew sym} \\ \text{other} & \text{nothing} \end{cases}$$

(5)

Putting all terms back together, the weak statement is:

$$\boxed{\sum_e \int (\nabla \hat{T} k \nabla T) dv} \ (1) \longrightarrow B^e_i$$

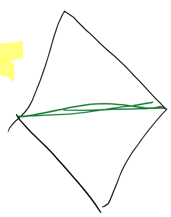


$$+ \sum_e \int_{\partial e_u} (\hat{T} q - \varepsilon T \hat{q}) ds \quad (2)$$

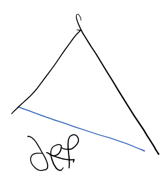
essential ∂e_u

$$+ \sum_{\Gamma^{e\bar{e}t}} \int [\hat{T}] (\{q\} + \alpha [T]) ds + \varepsilon \int \{\hat{q}\} [T] ds$$

(a) (c) (b) $\Gamma^{e\bar{e}t}$



$$= \boxed{\sum_e \int \hat{T} Q \, dv} \ (3) \quad + \sum \varepsilon \int_{\partial e_u} q \cdot n \, \hat{T} ds \quad - \int_{\partial e_f} \hat{T} \bar{q}_n \, ds$$



$\square$ CFEM terms   Essential BC terms   interior penalty (IP) terms

average of v at the endpoints of I_n:

$$[v(x_n)] = v(x_n^-) - v(x_n^+), \quad \{v(x_n)\} = \frac{1}{2}(v(x_n^-) + v(x_n^+)) \quad \forall n = 1, \ldots, N-1.$$

... average at the endpoints of t[illegible]

$$[v(x_n)] = v(x_n^-) - v(x_n^+), \quad \{v(x_n)\} = \frac{1}{2}(v(x_n^-) + v(x_n^+))$$

... average at the endpoints of t...

$$\forall x \in (0,1), \quad -(K(x)p'(x))' = f(x),$$
$$p(0) = 1,$$
$$p(1) = 0,$$

P solution $\rightarrow T$ v weight ($\hat{T}$)

We now note that the exact solution ... see that ~~if p is a solution of (1.1)–(1.3), then~~ p satisfies

$$\sum_{n=0}^{N-1}\int_{x_n}^{x_{n+1}} K(x)p'(x)v'(x)dx - \sum_{n=0}^{N}\{K(x_n)p'(x_n)\}[v(x_n)] + \epsilon\sum_{n=0}^{N}\{K(x_n)v'(x_n)\}[p(x_n)]$$
$$= \int_0^1 f(x)v(x)dx - \epsilon K(x_0)v'(x_0)p(x_0) + \epsilon K(x_N)v'(x_N)p(x_N)$$
$$= \int_0^1 f(x)v(x)dx - \epsilon K(x_0)v'(x_0).$$

1 a 3 $\{q\}[T]$ $\{\hat{q}\}[T]$

2D 3D $\hat{}$ $[q][q]$ $[\hat{T}][T']$

We can also add the following terms to the weak statement:
Optional terms

~~...rred to as penalty terms:~~

$$J_0(v,w) = \sum_{n=0}^{N}\frac{\sigma^0}{h_{n-1,n}}[v(x_n)][w(x_n)], \quad J_1(v,w) = \sum_{n=1}^{N-1}\frac{\sigma^1}{h_{n-1,n}}[v'(x_n)][w'(x_n)].$$

~~We note that J_0 penalizes the j...~~

αK weight sln $\hat{T}$ $[T]$ $[T]$ α c d

$$\boxed{\begin{aligned} q^* &= \{\{q\}\} + \vec{\beta}\,[[q]] + K\alpha[[T]] \\ T^* &= \{\{T\}\} + \vec{\delta}\cdot[[T]] + \beta[q] \end{aligned}}$$

$\rightarrow$ created term (c)

$\rightarrow$ would have created (d)

J1 term is rarely introduced in interior penalty methods, but J0 is often added (more discussion below)

J_1 α sufficiently large number

- If $\epsilon = -1$, $\sigma^1 = 0$, and σ^0 is bounded below by a large enough constant, the resulting method is called the symmetric interior penalty Galerkin (SIPG) method, introduced in the late 1970s by Wheeler [109] and Arnold [1].

symm. K is sym.

- If $\epsilon = -1$ and $\sigma^0 = \sigma^1 = 0$, the resulting method is called the global element method, introduced in 1979 by Delves and Hall [43]. However, the matrix associated with the bilinear form is indefinite, as the real parts of the eigenvalues are not all positive and thus the method is not stable.

$\rightarrow$ real part of eigenvalues of K become negative

Remedy large enough α term added (shifts real part of eigenvalues to $\mathbb{R}^+$)

$\alpha \neq 0$

- If $\epsilon = +1$, $\sigma^1 = 0$, and $\sigma^0 = 1$, the resulting method is called the nonsymmetric interior penalty Galerkin (NIPG) method, introduced in 1999 by Rivière, Wheeler, and Girault [95].

DG formulation I did in first sessions

- If $\epsilon = +1$ and $\sigma^0 = \sigma^1 = 0$, the resulting method was introduced by Oden, Babuška, and Baumann in 1998 [84]. Throughout these notes, we will refer to this method as the NIPG 0 method, since it corresponds to the particular case of NIPG with $\sigma^0 = 0$.

Coercive bilinear form

- If $\epsilon = 0$, we obtain the incomplete interior penalty Galerkin (IIPG) method introduced by Dawson, Sun, and Wheeler [42] in 2004.

$$+\,\varepsilon\int_{\partial e} \hat{q}\cdot(T^* - T)\,ds$$

enforces continuity of T & essential Br...

this term indirectly enforces continuity of T

jul α

$$q^* = \{\{q\}\} \cdot \vec{\beta}\,[[q]] + K\alpha[[T]]$$

enforces continuity of

$$\boxed{\begin{aligned} q^* &= \{\{q\}\} + \vec{\beta}\,[[q]] + K\alpha[[T]] \\ T^* &= \{\{T\}\} + \vec{\gamma}\cdot[[T]] \end{aligned}}$$

enforces continuity of T & essential BC

from $K\alpha[T]$ term

$\hat{T}\cdot(q^* - q) \longrightarrow \hat{T}\,[T]$

IP $\int \hat{T}\,[T]$ Indirect control of $[T]$

$\varepsilon \int \hat{q}\,(\hat{T} - T)$ on interior interfaces

$\hat{q}\,[T]$ direct ($\varepsilon = 1$

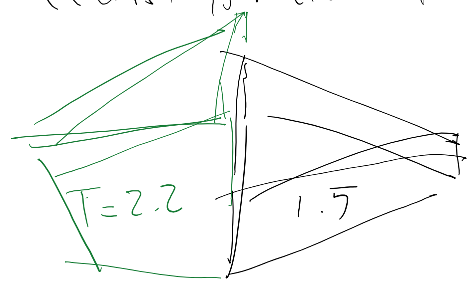
$\hat{q} = -K\nabla\hat{T} = 0$ Oden's method)

for $p=0$ elements (const per element)

with ~~but~~ possible

NIPG ($\alpha \neq 0$)

$p = 0$ is not feasible with Oden's method NIPG0



The discussion above was from Riviere_PenaltyMethod_formulation.pdf

Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations

Arnold_2002_Unified analysis of discontinuous Galerkin methods for elliptic problems.pdf

TABLE 6.1
Properties of the DG methods

very similar

Method	cons.	a.c.	stab.	type	cond.	H^1	L^2
Brezzi et al. [18]	✓	✓	✓	α^r	$\eta_0 > 0$	h^p	h^{p+1}
LDG [35]	✓	✓	✓	α^j	$\eta_0 > 0$	h^p	h^{p+1}
IP [43]	✓	✓	✓	α^j	$\eta_0 > \eta^*$	h^p	h^{p+1}
Bassi et al. [10]	✓	✓	✓	α^r	$\eta_0 > 3$	h^p	h^{p+1}
NIPG [53]	✓	×	✓	α^j	$\eta_0 > 0$	h^p	h^p
Babuška–Zlámal [6]	×	×	✓	α^j	$\eta_0 \approx h^{-2p}$	h^p	h^{p+1}

SIPG ($\varepsilon = -1$) Wheeler 1970s method

NIPG ($\varepsilon = 1$) all we need is

NIPG [53]	✓	×	✓	α^j	$\eta_0 > 0$	h^p	h^p
Babuška Zlámal [6]	×	×	✓	α^j	$\eta_0 \approx h^{-2p}$	h^p	h^{p+1}
Brezzi et al. [19]	×	×	✓	α^r	$\eta_0 \approx h^{-2p}$	h^p	h^{p+1}
Baumann Oden ($p = 1$)	✓	×	×	-	-	×	×
Baumann Oden ($p \geq 2$)	✓	×	×	-	-	h^p	h^p
Bassi Rebay [9]	✓	✓	×	-	-	$[h^p]$	$[h^{p+1}]$

NIPG ($\varepsilon = 1$) meaning(?)
all we need is
$\bar{\alpha} > 0$

NIPG $\bar{\alpha} = 0$ (? why p=1 is also back?)

look at paper for diff in α^r, α^j (what we basically did)

For Elliptic PDEs:

$\varepsilon = -1 \equiv$ SIPG need large $\bar{\alpha}$

$\varepsilon = 1$ NIPG $\sim$ LDG...

for good properties $\bar{\alpha} > 0$

we'll see for parabolic PDEs $\bar{\alpha}$ can be zero
(Oden method is fine for $p > 0$?