DG2020/02/17

Monday, February 17, 2020 11:41 AM

Discussion of the coercivity of the bilinear form:

From the last time we had:

$$B(\hat{T}, T) = L(\hat{T})$$

$$\sum_e \int_e \nabla \hat{T} \, k \nabla T \, dv + \sum_e \int_{\partial e_N} \left(\hat{T} q - \varepsilon T \hat{q} \right) \cdot n \, ds$$

$$+ \sum_{\Gamma_{ee'}} \int_{\Gamma_{ee'}} \left\{ \left([\hat{T}] \{q\} - \varepsilon [T] \{\hat{q}\} \right) + \bar{\alpha} [\hat{T}][T] \right\} ds = \sum L_i^e(\hat{T}) + \sum_{\partial e_u} L_u^e(\hat{T}) + \sum_{\partial e_f} L_f^e(\hat{T})$$

$$\boxed{\hat{T} = T}$$ plug this in bilinear statement to get:

$$\boxed{\begin{aligned} B(T, T) &= \int_e \nabla T K \nabla T \, dv + \sum_e \int \left(T q - \varepsilon T q \right) \cdot n \, ds \\ &\quad + \sum_{\Gamma_{ee'}} \int \left([T]\{q\} - \varepsilon [T]\{q\} \right) + \bar{\alpha} [T][T] \end{aligned}}$$

$\varepsilon = 1$ nonsymmetric (NIPG) $\longrightarrow$

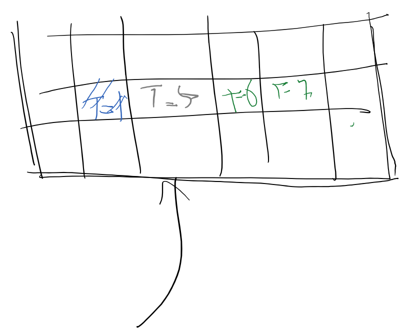
$$\boxed{B(T, T) = \underbrace{\int_e \nabla T \, \underset{\text{sym matrix} > 0}{k} \, \nabla T \, dv}_{\geq 0} + \sum_{\Gamma_{ee'}} \underbrace{\int_{\Gamma_{ee'}} \alpha [T][T]}_{\geq 0} \geq 0}$$

what if $\bar{\alpha} = 0$ is $B(T, T) > 0$ for $T \neq 0$

$B(T, T) = 0$ for global $T \neq 0$

$T = 0$

∂D



if we add the $\bar{\alpha}$ term ($\bar{\alpha} \neq 0$)

$$B(T, T) = 0 \Longrightarrow T = 0$$

The condition B(T, T) >= lamba |T|

Coercivity of bilinear form.

$$\boxed{\exists \lambda > 0 \ni \forall T \quad B(T, T) \geq \lambda \underbrace{|T|^2}_{L^2 \text{ norm of Temperature field}}}$$

$$\boxed{\exists \lambda > 0 \ \ni \forall T \quad B(T,T) \geq \lambda \underbrace{|T|^2}_{\substack{L^2 \text{ norm of} \\ \text{Temperature field}}}}$$

For $\lambda > 0$ $B(T,T)$ is coercive for $\varepsilon = 1$ (NIPG) & one can prove the stability of the method.

Coercivity is in general an integral part of stability proofs for elliptic PDEs.
Outline …

T_1 & T_2 are s'lns

$$\left.\begin{array}{ll} T_1 \text{ sln} & \forall \tilde{T} \in V \ \ B(\tilde{T}, T_1) = L(\tilde{T}) \\ T_2 \text{ sln} & \forall \tilde{T} \in V \ \ B(\tilde{T}, T_2) = L(\tilde{T}) \end{array}\right\} \text{ subtract}$$

$$\forall \tilde{T} \quad B(\tilde{T}, T_1 - T_2) = 0 \qquad \text{choose } \hat{T} = T_1 - T_2 \in V$$

$$\longrightarrow \quad B(T_1 - T_2, T_1 - T_2) = 0 \qquad \text{if we have coercivity}$$

$$\lambda |T_1 - T_2|^2 \leq B(T_1 - T_2, T_1 - T_2) = 0$$

$$\Longrightarrow \quad \lambda |T_1 - T_2| = 0 \quad \Longrightarrow \quad \boxed{T_1 = T_2}$$

$Kx = F$ $\quad \det K \neq 0 \longrightarrow$ proves uniqueness $x = K^{-1} F$
but for stability that's where we need coercivity

What is the relation of coercivity and system matrix?

$$B(\hat{T}, T) = \hat{a} K a$$

discrete form of this

$\hat{a}$: vector for the weights

a: unknown vector

$$B(T, T) = a . K a$$

for coercivity we want

$$B(T, T) \geq \lambda |T^2| \quad \equiv \quad \boxed{a . K a \geq \lambda |a|^2}$$

$\lambda \geq 0$

a arbitrary

stiffness matrix

if K has a $\lambda_i \le 0$

$Ku = \lambda_i u \qquad u \cdot Ku = \lambda_i |u|^2 \le 0 \qquad K$ is not coercive

So if All eigenvalues of K are positive K is coercive

& $\lambda = \min\{\lambda_i\} \qquad a \cdot Ka \ge \lambda |a|^2 \quad \forall a$

Coercivity of the bilinear form is equivalent to the positive-definiteness of the discrete stiffness matrix and lambda (lower bound of coercivity) corresponds to the smallest positive eigenvalue of K

This is why the NIPG is stable.

How about SIPG:

$\varepsilon = -1$

IIPG $\varepsilon = 0$

$$B(T,T) = \overbrace{B_{\varepsilon=1}(T,T)}^{\ge 0} + f\left(\sum_{\partial e_4} \int T q \cdot n \, ds + \int \sum_{e\partial t} [T]\{q\}\, ds\right)$$

$$f = \begin{cases} 1 & \varepsilon = 0 \\ 2 & \varepsilon = -1 \end{cases}$$

large enough

$T K \nabla T$

$$B(T,T) = \underbrace{\int_e \nabla T \underset{\text{sym} > 0 \text{ matrix}}{K} \nabla T \, dv}_{\ge 0} + \underbrace{\sum_{\Gamma_{e\partial t}} \int_{\Gamma_{e\partial t}} \bar{\alpha} [T][T]}_{\ge 0} + f\left(\sum_{\partial e_4} \int T q \cdot n \, ds + \sum_{\Gamma_{e\partial t}} \int [T]\{q\}\, ds\right)$$

$\rightarrow B(T,T)$ becomes coercive

$$-3 + 5\bar{\alpha} \ge 0 \qquad \bar{\alpha} \ge \frac{3}{5}$$

- If $\epsilon = -1$, $\sigma^1 = 0$, and σ^0 is bounded below by a large enough constant, the resulting method is called the symmetric interior penalty Galerkin (SIPG) method, introduced in the late 1970s by Wheeler [109] and Arnold [1].
- If $\epsilon = -1$ and $\sigma^0 = \sigma^1 = 0$, the resulting method is called the global element method, introduced in 1979 by Delves and Hall [43]. However, the matrix associated with the bilinear form is indefinite, as the real parts of the eigenvalues are not all positive and thus the method is not stable.
- If $\epsilon = +1$, $\sigma^1 = 0$, and $\sigma^0 = 1$, the resulting method is called the nonsymmetric interior penalty Galerkin (NIPG) method, introduced in 1999 by Rivière, Wheeler, and Girault [95].
- If $\epsilon = +1$ and $\sigma^0 = \sigma^1 = 0$, the resulting method was introduced by Oden, Babuška, and Baumann in 1998 [84]. Throughout these notes, we will refer to this method as the NIPG 0 method, since it corresponds to the particular case of NIPG with $\sigma^0 = 0$.
- If $\epsilon = 0$, we obtain the incomplete interior penalty Galerkin (IIPG) method introduced by Dawson, Sun, and Wheeler [42] in 2004.

Method	Cons.	A.C.	Stab.	Type	Cond.	H^1	L^2
Brezzi et al. [22]	✓	✓	✓	α^r	$\eta_0 > 0$	h^p	h^{p+1}
LDG [41]	✓	✓	✓	α^j	$\eta_0 > 0$	h^p	h^{p+1}
IP [50]	✓	✓	✓	α^j	$\eta_0 > \eta^*$	h^p	h^{p+1}

Brezzi et al. [22]	✓	✓	✓	α^r	$\eta_0 > 0$	h^p	h^{p+1}
LDG [41]	✓	✓	✓	α^j	$\eta_0 > 0$	h^p	h^{p+1}
IP [50]	✓	✓	✓	α^j	$\eta_0 > \eta^*$	h^p	h^{p+1}
Bassi et al. [13]	✓	✓	✓	α^r	$\eta_0 > 3$	h^p	h^{p+1}
NIPG [64]	✓	×	✓	α^j	$\eta_0 > 0$	h^p	h^p
Babuška–Zlámal [7]	×	×	✓	α^j	$\eta_0 \approx h^{-2p}$	h^p	h^{p+1}
Brezzi et al. [23]	×	×	✓	α^r	$\eta_0 \approx h^{-2p}$	h^p	h^{p+1}
Baumann–Oden ($p = 1$)	✓	×	×	-	-	×	×
Baumann–Oden ($p \geq 2$)	✓	×	×	-	-	h^p	h^p
Bassi–Rebay [10]	✓	✓	×	-	-	$[h^p]$	$[h^{p+1}]$

~~(Baumann, p >= 2)~~

α $\left(\bar{\alpha} = \frac{\alpha}{h}\right)$

$\alpha > 0$

Material needed for next HW (elliptic PDE code example)

1D

$\{T\} = \frac{T^- + T^+}{2}$

$[T] = T^- n^- + T^+ n^+ = T^-(1) + T^+(-1)$
$= T^- - T^+$

$\{q\} = \frac{q^- + q^+}{2}$

$[q] = q^- \cdot n^- + q^+ n^+ = q^- - q^+$

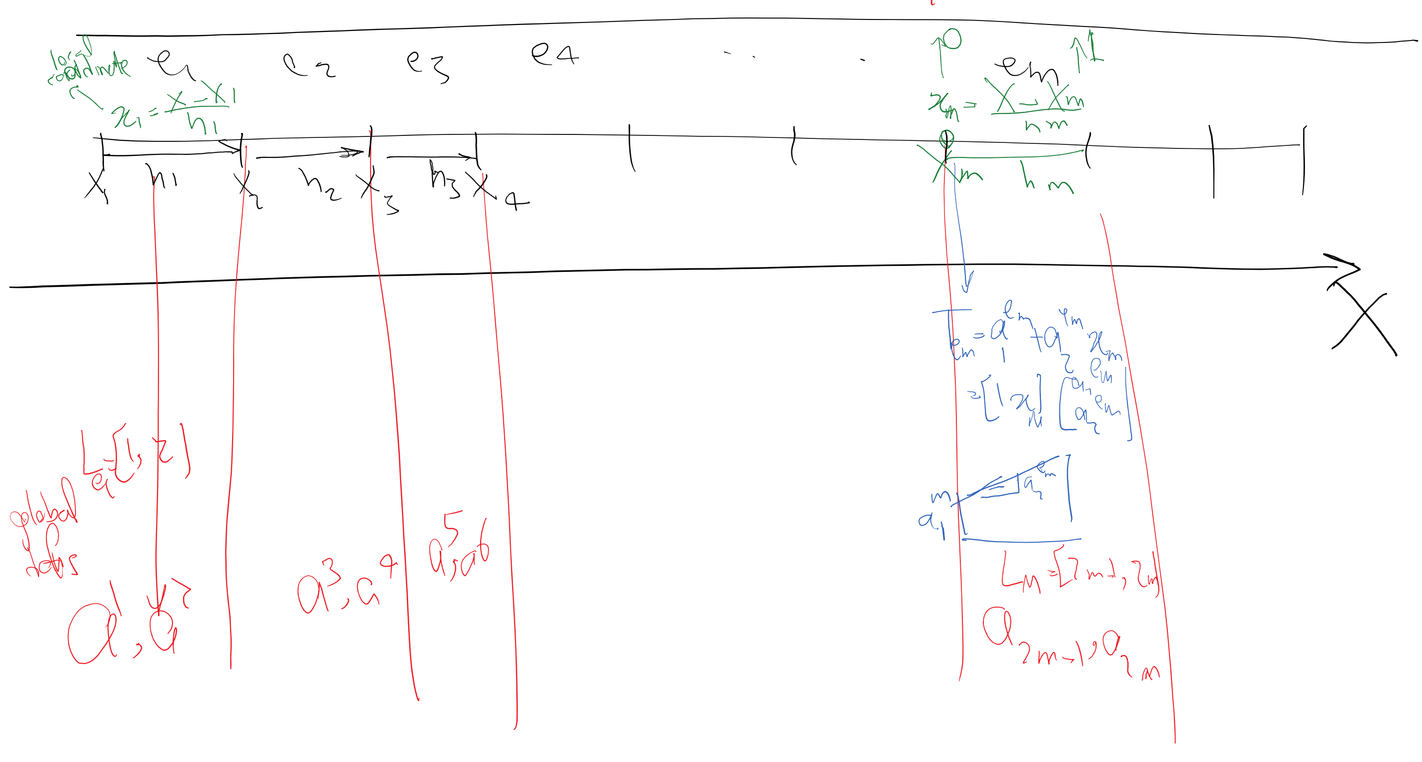
$-$ $\quad$ $+$

T^- $\quad$ $n^+ = -1$ $\quad$ T^+

q^- $\quad$ $n^- = 1$ $\quad$ q^+

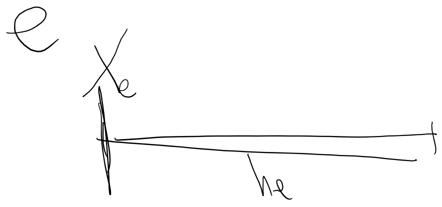
1D $\boxed{[\phi] = \phi^- - \phi^+}$

left right



Now for an arbitrary element, we form its contribution to global K, F from Essential BC, Natural BC, and interior of the element.

Also, we form contributions from the interior interfaces again to global K and F



$$\boxed{T = [1 \quad x]\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}} = \left[1 \; , \; \frac{X - X_e}{h}\right]\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$x = \frac{X - X_e}{h_e}$$

$$dx = \frac{dX}{h_e}$$

$$T' = \frac{dT}{dX} \longrightarrow \boxed{T' = \left[0 \, , \, \frac{1}{h}\right]\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}}$$

$$q = -k T' \longrightarrow \boxed{q = \left[0 \, , \, -\frac{k}{h}\right]\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}}$$

A. Interior of element contributions:

$$K^e_{int} = \int_e \nabla \hat{T} k \nabla T \, de = \int_{X_e}^{X_e + h_e} \hat{T}' k T' \, dX$$

$$= \int_0^1 \hat{T}' k T' (h_e dx) = \int_0^1 \begin{bmatrix} 0 \\ \frac{1}{h_e} \end{bmatrix} k \left[1 \;\; \frac{1}{h_e}\right] h_e \, dx$$

$$\boxed{K^e_{int} = \frac{k}{h_e}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \quad \text{(1a)}$$

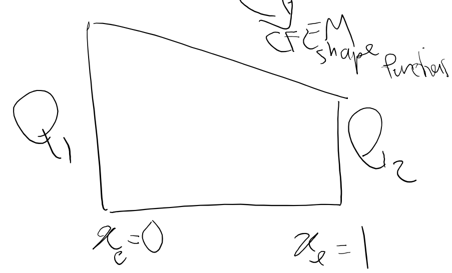
if we want to solve Parabolic PDE $\; C\dot{T} + \nabla \cdot q = Q$

if this exists

$$M = \int_e \hat{T} C T \, dv = \int_0^1 \begin{bmatrix} 1 \\ x \end{bmatrix} C [1 \;\; x](h \, dx)$$

$$\boxed{m^e = Ch\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix}} \quad \text{(1b)}$$

$$Q_e = (1-x)Q_1 + xQ_2$$



Source term:

$$f^e_Q = \int \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} Q(x) \, dX$$

$$= \int_0^1 \begin{bmatrix} 1 \\ x \end{bmatrix} \Big(\underbrace{[1-x \;\; x]\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}}_{Q(x)} \Big) (h \, dx)$$

$$\boxed{f^e_Q = h\begin{bmatrix} \frac{1}{2} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}} \quad \text{(1c)}$$

2. T-continuous Fr: $T = [1 \;\; x]$ $\quad 1 \;\; x$

2. Essential BC.

$$B_u(\tilde{T}, T) = \int_{\partial e_u} (\tilde{T} q - \varepsilon T \hat{q}) \cdot n \, ds$$

$$L_u(\tilde{T}) = -\varepsilon \int_{\partial e_u} \hat{q} \cdot n \, \bar{T} \, ds$$

$T = [1 \;\; x]$

$T' = [1 \;\; \tfrac{1}{h}]$

$q = [0 \;\; -\tfrac{k}{h}]$

x

0 1

$\partial e_u = \{$essential boundary of element$\}$

$$K_u = (\tilde{T} q - \varepsilon T \hat{q}) \cdot n \Big|_{@ \partial e_u} = \left(\begin{bmatrix} 1 \\ x \end{bmatrix} [0 \;\; -\tfrac{k}{h}] - \varepsilon \begin{bmatrix} 0 \\ -\tfrac{k}{h} \end{bmatrix} [1 \;\; x] \right) n$$

$$F_u = -\varepsilon \hat{q} \cdot n \, \bar{T} \, @ \, \partial e_u = -\varepsilon \begin{bmatrix} 0 \\ -\tfrac{k}{h} \end{bmatrix} \vec{n} \, \bar{T}$$

∂e_u $x = 0$

$n = -1$

$x = 0$ & $n = -1$

$$K_u^L = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} [0 \;\; -\tfrac{k}{h}] - \varepsilon [0 \;\; -\tfrac{k}{h}] \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) (-1)$$

simplify it

$$f_u^L = -\varepsilon (-1) \bar{T} \begin{bmatrix} 0 \\ -\tfrac{k}{h} \end{bmatrix}$$

dofs 1 & 2 global system

2

$x = 1$

$n = 1$

$$K_u^R = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} [0 \;\; -\tfrac{k}{h}] \ldots \right) (1)$$

$$f_u^R = -\varepsilon (1) \bar{T} \begin{bmatrix} 0 \\ -\tfrac{k}{h} \end{bmatrix}$$

dofs

$2N-1, 2N$

where we have N elements

3. Natural BC

$$\cancel{B_f(\tilde{T}, T)}$$

$$L_f(\tilde{T}) = \int_{\partial e_f} \tilde{T} \, \bar{q}_n \, ds \rightarrow F_f = \begin{bmatrix} 1 \\ x \end{bmatrix} \bar{q}_n \;\; @ \text{ natural BC}$$

$\bar{q}_n$ $x = 0$

$n = -1$

$$F_f = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \bar{q}_n$$

dofs 1, 2, ... system

x

$x = 1$ if natural

$n = 1$

$$2N-1 \leftarrow F_n = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \bar{q}_n$$

$T_R = [c]^{\psi}_{n}$

$\mathbb{R}^2$

global system

$$\begin{array}{r} 2N-1 \leftarrow \\ 2N \leftarrow \end{array} F_n = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \bar{q}_n$$