DG2020/02/19

Wednesday, February 19, 2020 11:41 AM

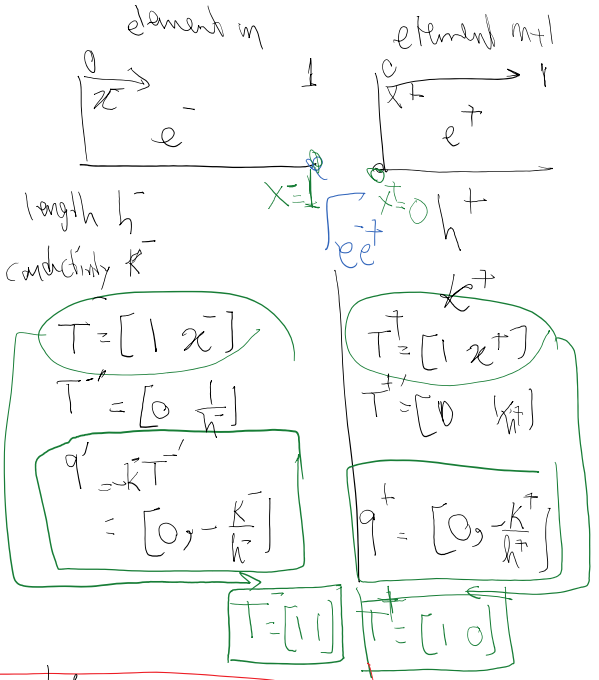
4) K and F terms for interior interfaces

$$\int_{e^-e^+} \left(\underbrace{[\hat{T}]\{q\}}_{a} + \underbrace{\alpha[T]}_{b} \right) - \varepsilon \{q\}[T] \, dx$$

$[T] \longrightarrow$ row vectors $[\tilde{T}]$

$\{q\} \longrightarrow =$ $\{q\}$

we want to form $K^L_{4\times 4}$ and later we can assemble it in the global system.



$$[\hat{T}] = \hat{T}^- - \hat{T}^+ = \begin{matrix} e^- \\ e^+ \end{matrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{matrix} e^- \\ e^+ \end{matrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

dofs $2m-1, 2m$

dofs $2m+1, 2m+2$

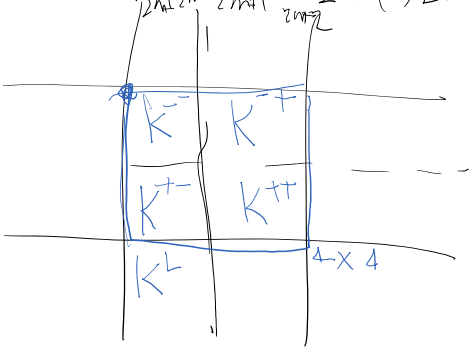
$$\{\hat{q}\} = \frac{(\hat{q}^- + \hat{q}^+)}{2} = \frac{1}{2}\left(\begin{bmatrix} 0 \\ -\frac{K^-}{h^-} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{K^+}{h^+} \end{bmatrix} \right) = \begin{bmatrix} 0 \\ -\frac{K^-}{2h^-} \\ 0 \\ -\frac{K^+}{2h^+} \end{bmatrix}$$

$$[T] = [\hat{T}]^t \qquad \{q\} = \{q\}^T$$

global K

$2m-1$
$2m$
$2m+1$
$2m+2$

weight



add them

(a) $[\hat{T}]\{q\} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & -\frac{K^-}{2h^-} & 0 & -\frac{K^+}{2h^+} \end{bmatrix}$

(b) $\alpha[\tilde{T}][T] = \alpha \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix}$

(c) $-\varepsilon\{\hat{q}\}[T] = -\varepsilon$

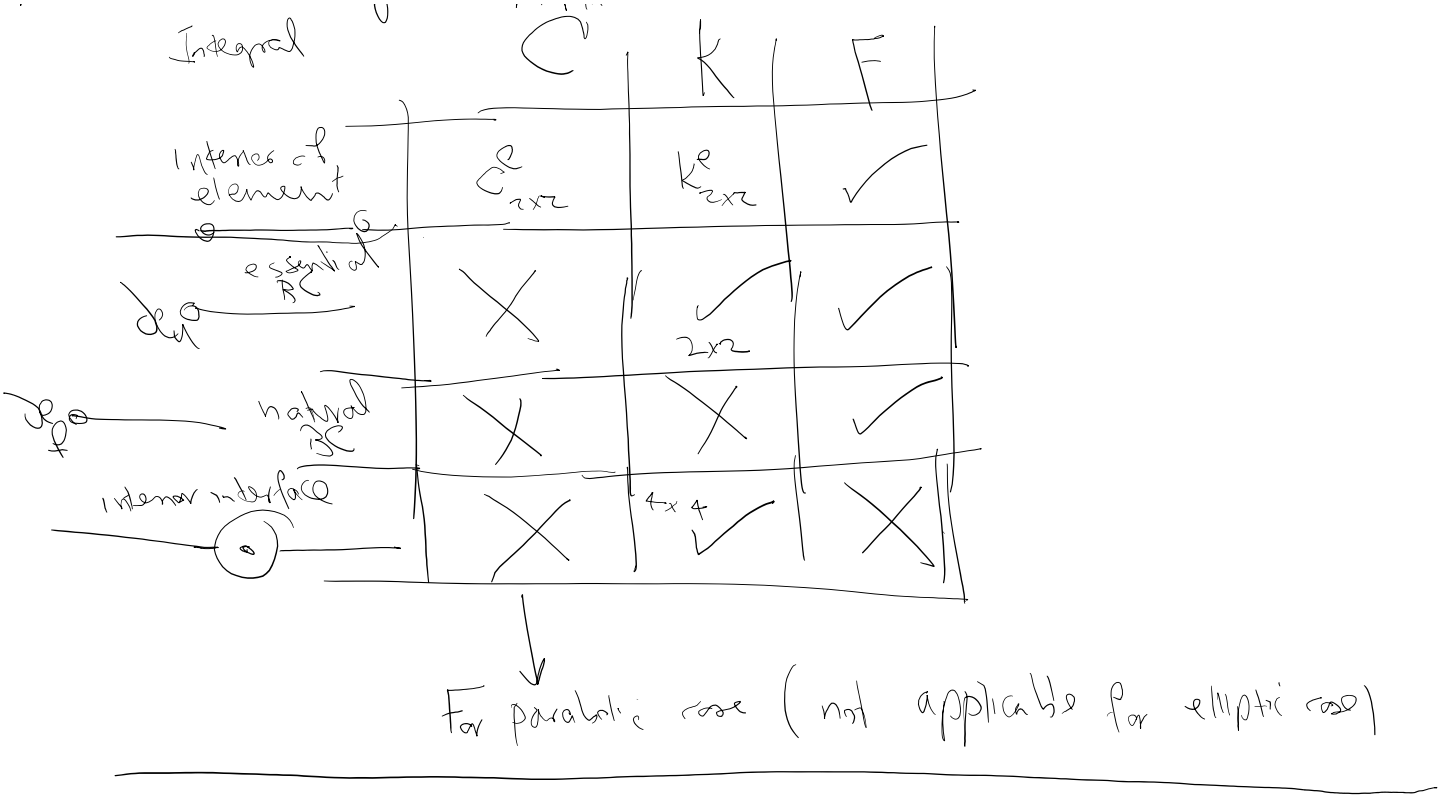
this $K^L_{4\times 4}$ corresponding to the interface between elements e^-, e^+

assemble $K^L_{4\times 4}$ in global K at dofs $(2m-1, 2m, 2m+1, 2m+2)$

After the assembly

Integral

Capacity matrix C | stiffness / conductivity K | force vector F L



Parabolic case

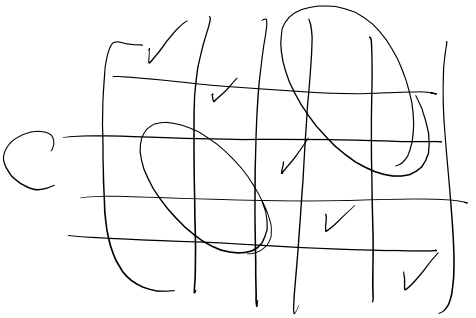
$$C\dot{a} + Ka = F$$

we'll use a forward in time FD scheme

$$C\left(\frac{a_{n+1} - a_n}{\Delta t}\right) + Ka_n = F_n$$

$$Ca_{n+1} = Ca_n - \Delta t Ka_n + F_n$$

save for a_{n+1}



In practice we don't form global C, and only solve this equation at the element level. Meaning that we advance the solution by 1 element at a time.

Again in practice, we don't form K for evaluating the RHS and Ka_n is computed at the element level.

For parabolic set exercises

$$\Delta t < \Delta t_{e_i}$$

diff eqn $\leftarrow D_i$, h_i

$$\Delta t_i = C h_i^2$$

$$\Delta t_i = C_p \frac{h_i^2}{2\nu_i}$$

Some points about elliptic and parabolic PDEs and their DG formulation

1. Single-field formulation versus multi-field formulation of the problem

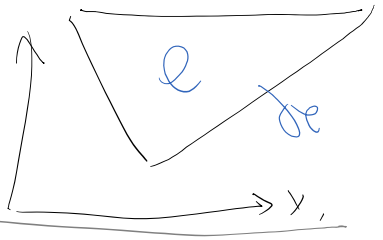
$\tilde{T}$ (a) PDE from balance law $\longleftarrow C\dot{T} + \nabla \cdot q = Q$

$\tilde{q}$ (b) const. eqn $\longleftarrow$ $K\nabla T = -q \rightarrow \nabla T = -K^{-1} q$

2-field formulation where both T & q are interpolated

$T = a_0 + a_1 x + a_2 y$

$q = \begin{bmatrix} q_x \\ q_y \end{bmatrix} = \begin{bmatrix} a_3 + a_4 x + a_5 y \\ a_6 + a_7 x + a_8 y \end{bmatrix}$



we need $\hat{T}$, $\hat{q}$

$$\int_e \hat{T}\left(C\dot{T} + \nabla \cdot q - Q\right) dv + \int_{\partial e} \hat{T}\left(q_n^* - q \cdot n\right) ds$$
$$-\varepsilon^* \left(-\int_e \hat{q}\left(\nabla T + K^{-1} q\right) dv - \int_{\partial e} \hat{q} \cdot n \left(T^* - T\right) ds \right) = 0 \quad (2F)$$

We can solve for T and q and independent fields. Assemble K and F as usual …

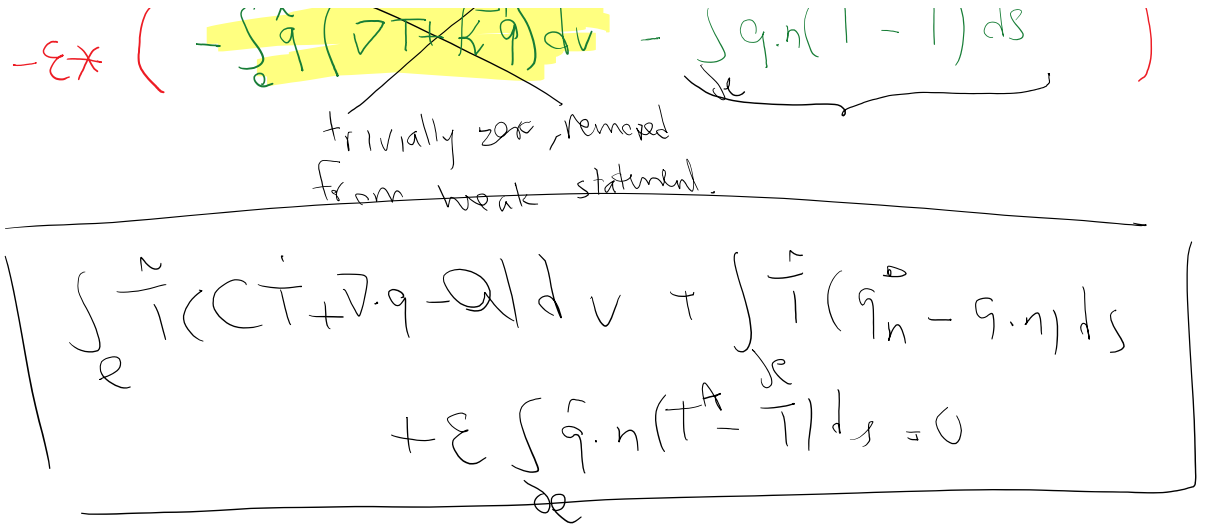
Ideas: How to get rid of q at the global level (global stiffness):

Idea 1: Don't interpolate q! If means we only have $T = a_0 + a_1 x + a_2 y$

$q = -K\nabla T$ is computed

plug this in (2F)

$$\int_e \hat{T}\left(C\dot{T} + \nabla \cdot q - Q\right) dv + \int_{\partial e} \hat{T}\left(q_n^* - q \cdot n\right) ds$$
$$-\varepsilon^* \left(-\int_e \hat{q}\left(\nabla T + K^{-1} q\right) dv - \int_{\partial e} \hat{q} \cdot n \left(T^* - T\right) ds \right) = 0$$



This is exactly the same approach we used in coming up with a 1F formulation for this problem and the integral in the second line was added to enforce continuity of T

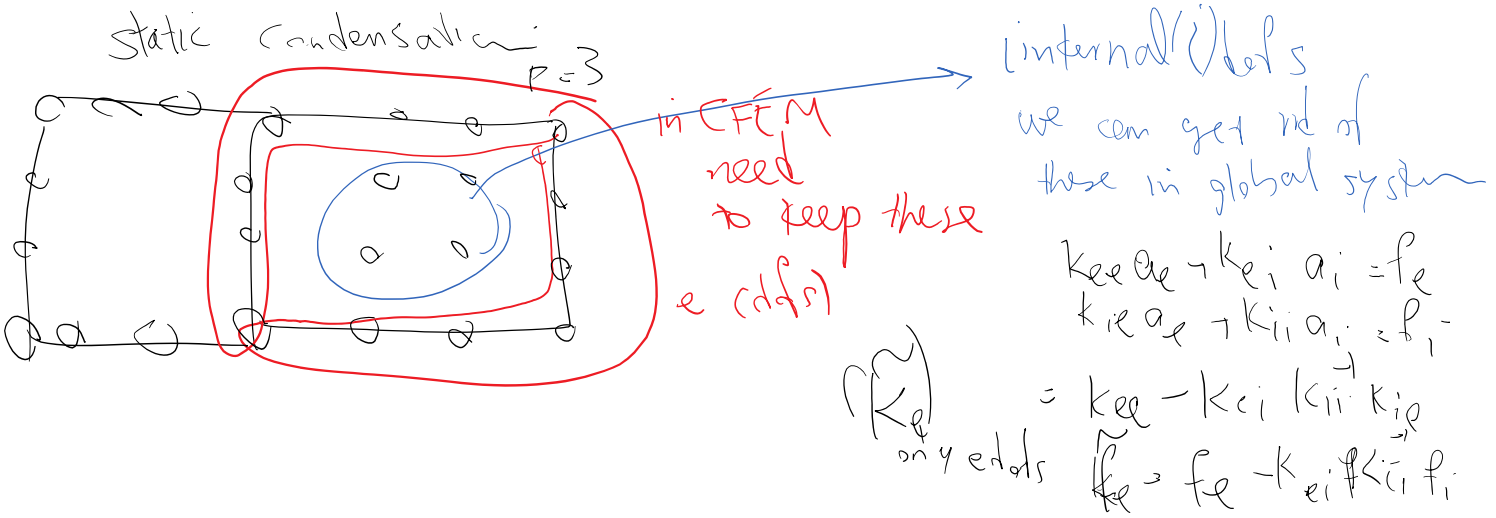
If eps = 0 we need to have [T] present in the formulation of q*

In 1-field formulation there is no independent q field. So, where ever we see q, basically it means

$$q = -k\nabla T$$

2. Removing q dofs from the global system

This is the second approach (first one, doing a 1F formulation discussed above) for removing q from global matrix dofs for 2F formulation



We can apply the same trick with DG formulation by eliminating dofs at the element level that can be eliminated from the global system

$T^* = \{T\} + \gamma[T]$

eliminate q in the global system (like bbe dof's above)

if not have this term

what if $+\beta[q]$

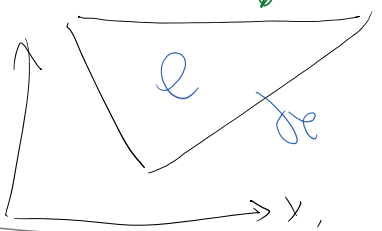
if we add this term we cannot get rid of element level T OR q fields in the global system

(no static condensation is possible for global K)

all q dofs go to global K

$T = a_0 + a_1 x + a_2 y$

$q = \begin{bmatrix} q_x \\ q_y \end{bmatrix} = \begin{bmatrix} a_3 + a_4 x + a_5 y \\ a_6 + a_7 x + a_8 y \end{bmatrix}$



we need $\hat{T}$, $\hat{q}$

$$\int_e \hat{T}(C\dot{T} + \nabla\cdot q - Q)\,dv + \int_{\partial e} \hat{T}(q_n^* - q\cdot n)\,ds$$

$$-\xi^* \left(-\int_e \hat{q}(\nabla T + k^{-1} q)\,dv - \int_{\partial e} \hat{q}\cdot n (T^* - T)\,ds \right) = 0$$

(25)

T

Arnold 2002

Finally, it is important to note that in all the methods we are going to analyze, $h_u^{e,K}$ will not depend on $\sigma_h|_{K_i}$ (nor on $\nabla u_h|_{K_i}$, but that will be less important). This, as we shall see, will allow us to eliminate the variable σ_h at the element level, often with a considerable computational saving.

q

The trick is the use of

Recalling that $\nabla_h V_h \subset \Sigma_h$ and defining lifting operators $r : [L^2(\Gamma)]^2 \to \Sigma_h$ and $l : L^2(\Gamma^0) \to \Sigma_h$ by

$$(3.8)\quad \int_\Omega r(\varphi)\cdot\tau\,dx = -\int_\Gamma \varphi\cdot\{\tau\}\,ds, \qquad \int_\Omega l(q)\cdot\tau\,dx = -\int_{\Gamma^0} q[\![\tau]\!]\,ds \quad \forall \tau\in\Sigma_h,$$

these operators

Questions you may have: Why even bother using a 2F formulation and then try to condense q dofs out rather than simply using a 1F formulation? a) We can interpolate T and q both with order p rather than having order p and p - 1 for T and 1 in DG formulation with simplicial elements.

b)

TABLE 3.1
Some DG methods and their numerical fluxes.

Method	$\widehat{u}_K$	$\widehat{\sigma}_K$
Bassi–Rebay [10]	$\{u_h\}$	$\{\sigma_h\}$
Brezzi et al. [22]	$\{u_h\}$	$\{\sigma_h\} - \alpha_r([\![u_h]\!])$
LDG [41]	$\{u_h\} - \beta \cdot [\![u_h]\!]$	$\{\sigma_h\} + \beta[\![\sigma_h]\!] - \alpha_j([\![u_h]\!])$
IP [50]	$\{u_h\}$	$\{\nabla_h u_h\} - \alpha_j([\![u_h]\!])$
Bassi et al. [13]	$\{u_h\}$	$\{\nabla_h u_h\} - \alpha_r([\![u_h]\!])$
Baumann–Oden [15]	$\{u_h\} + n_K \cdot [\![u_h]\!]$	$\{\nabla_h u_h\}$
NIPG [64]	$\{u_h\} + n_K \cdot [\![u_h]\!]$	$\{\nabla_h u_h\} - \alpha_j([\![u_h]\!])$
Babuška–Zlámal [7]	$(u_h\|_K)\|_{\partial K}$	$-\alpha_j([\![u_h]\!])$
Brezzi et al. [23]	$(u_h\|_K)\|_{\partial K}$	$-\alpha_r([\![u_h]\!])$

TABLE 6.1
Properties of the DG methods.

Method	Cons.	A.C.	Stab.	Type	Cond.	H^1	L^2
Brezzi et al. [22]	✓	✓	✓	α^r	$\eta_0 > 0$	h^p	h^{p+1}
LDG [41]	✓	✓	✓	α^j	$\eta_0 > 0$	h^p	h^{p+1}
IP [50]	✓	✓	✓	α^j	$\eta_0 > \eta^*$	h^p	h^{p+1}
Bassi et al. [13]	✓	✓	✓	α^r	$\eta_0 > 3$	h^p	h^{p+1}
NIPG [64]	✓	×	✓	α^j	$\eta_0 > 0$	h^p	h^p
Babuška–Zlámal [7]	×	×	✓	α^j	$\eta_0 \approx h^{-2p}$	h^p	h^{p+1}
Brezzi et al. [23]	×	×	✓	α^r	$\eta_0 \approx h^{-2p}$	h^p	h^{p+1}
Baumann–Oden ($p = 1$)	✓	×	×	-	-	×	×
Baumann–Oden ($p \geq 2$)	✓	×	×	-	-	h^p	h^p
Bassi–Rebay [10]	✓	✓	×	-	-	$[h^p]$	$[h^{p+1}]$

But weakly stable (Baumann, p >= 2)

Relatively recently HDG (hybridizable DG) is proposed where this process of getting rid of certain dofs in the global system becomes much easier (like CFEM) and there is no need to l and r operators of LDG method.