DG2020/02/24

Monday, February 24, 2020   11:38 AM

Discussion points:
Read Arnold 2002 on how breaking the self-adjointness (breaking the symmetry of K matrix) for NIPG and Oden-Babuska method (which is NIPG with zero alpha) affects their convergence rate (no longer optimal convergence rate)

Optimal convergence rate in L2 norm for a field that is interpolated with order p is p + 1

T_h is order p $L_2(T_h - T_{exact}) = C h^{p+1}$

In this section we show that for the pure penalty methods and NIPG we can choose the penalty large enough to reduce the consistency error to the point where it does not interfere with optimal order convergence. We achieve this by choosing the penalty parameter η_e proportional to a negative power of h_e instead of keeping it bounded as for the consistent methods. However, this superpenalty procedure tends to make the DG method behave like a standard conforming method and thus significantly increases the condition number of the stiffness matrix.

3. Effect of penalty term on condition number
   In general adding large penalty values to a formulation (displacement continuity, contact models, …) we get bad stiffness conditioning

PERFORMANCE OF DISCONTINUOUS GALERKIN METHODS FOR ELLIPTIC PDE'S

PAUL CASTILLO *

Abstract. In this paper, we compare the performance of the main discontinuous Galerkin (DG) methods for elliptic partial differential equations on a model problem. Theoretical estimates of the condition number of the stiffness matrix are given for DG methods whose bilinear form is symmetric, which are shown to be sharp numerically. Then, the efficiency of the methods in the computation of both the potential and its gradient is tested on unstructured triangular meshes.

actual penalty $\bar{\alpha} = \frac{\alpha}{h}$

method	penalization	$\kappa(h)$
Babuška-Zlámal	$O\left(h^{-(2p+1)}\right)$	$O\left(h^{-(2p+2)}\right)$
IP	$O\left(h^{-1}\right)$	$O\left(h^{-2}\right)$
LDG	$O\left(h^{-1}\right)$	$O\left(h^{-2}\right)$
Baumann-Oden	no penalization	$O\left(h^{-2}\right)$
NIPG1	$O\left(h^{-1}\right)$	$O\left(h^{-2}\right)$
NIPG3	$O\left(h^{-3}\right)$	$O\left(h^{-4}\right)$

Arnold 2002

TABLE 6.1
Properties of the DG methods.

α

Method	Cons.	A.C.	Stab.	Type	Cond.	H^1	L^2
Brezzi et al. [22]	✓	✓	✓	α^r	$\eta_0 > 0$	h^p	h^{p+1}
LDG [41]	✓	✓	✓	α^j	$\eta_0 > 0$	h^p	h^{p+1}
IP [50]	✓	✓	✓	α^j	$\eta_0 > \eta^*$	h^p	h^{p+1}
Bassi et al. [13]	✓	✓	✓	α^r	$\eta_0 > 3$	h^p	h^{p+1}
NIPG [64]	✓	×	✓	α^j	$\eta_0 > 0$	h^p	h^p
Babuška–Zlámal [7]	×	×	✓	α^j	$\eta_0 \approx h^{-2p}$	h^p	h^{p+1}
Brezzi et al. [23]	×	×	✓	α^r	$\eta_0 \approx h^{-2p}$	h^p	h^{p+1}
Baumann–Oden ($p = 1$)	✓	×	×	-	-	×	×
Baumann–Oden ($p \geq 2$)	✓	×	×	-	-	h^p	h^p
Bassi–Rebay [10]	✓	✓	×	-	-	$[h^p]$	$[h^{p+1}]$

suboptimal

convergence rate

4. From this discussion, it appears that LDG method is the best, because it just needs a penalty that scales with 1/h and the fa ctor of it (eta_0 in table above) only needs to be positive.

The downside of LDG method is that it's a 2-field formulation (T, q), but as I discussed last time by using lift operators l, r and using the fact that T* is only a function of T we can condense q out from the global system.

$$\tilde{\alpha} = \frac{\alpha}{h}$$

What about beta and gamma terms in LDG method:

$$q^* = \{q\} + \vec{\beta}[q] + \tilde{\alpha}[T]$$
$$T^* = \{T\} + \vec{\gamma}[T]$$

What are beta and gamma? What choices we take when they are nonzero? We already analyzed the LDG method and related it to IP methods for beta = gamma = 0.

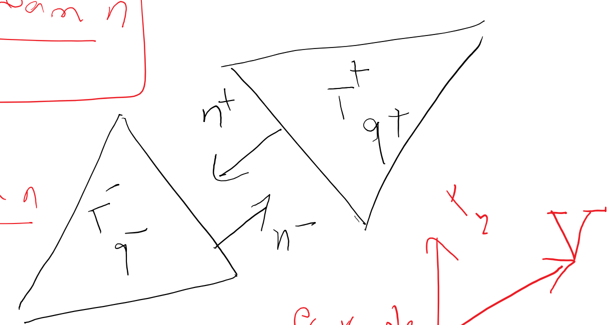
$\beta = \frac{n^-}{2}$ if $\underbrace{n^- \cdot V \geq 0}_{\text{like the figure}}$

else $\beta = \frac{n^+}{2}$

β ~~is~~ $\frac{\text{upstream } n}{2}$

γ is $\frac{\text{downsteam } n}{2}$

$\gamma = -\beta$



for example choose $\vec{V} = (2,1)$

$$q^* = \{q\} + \vec{\beta}[q] + \tilde{\alpha}[T]$$
$$T^* = \{T\} + \vec{\gamma}[T]$$

for the figure where $\beta = \frac{n^-}{2}$

$$q^* = \left\{\frac{q^- + q^+}{2}\right\} + \frac{n^-}{2}\left\{q^- n^- + q^+ n^+\right\} + \tilde{\alpha}[T] = \frac{q^- + q^+}{2} + \frac{q^- n^- \cdot n^-}{2} + \frac{q^+ n^- \cdot n^+}{2} + \tilde{\alpha}[T] =$$

$$q^* = \frac{q^- + q^+}{2} + \frac{q^-}{2} - \frac{q^+}{2} + \tilde{\alpha}[T] = q^- + \alpha[T]$$

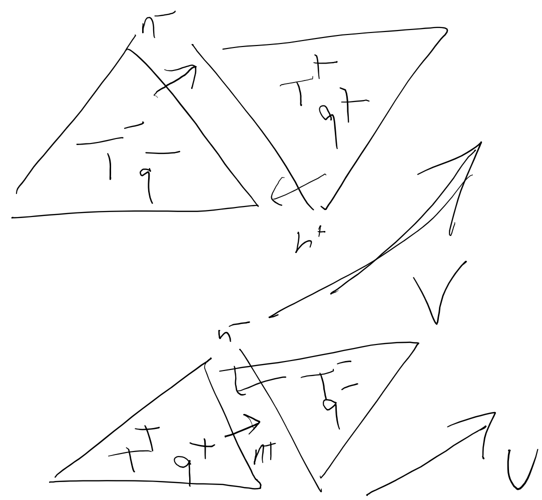
$$T^* = \frac{T^- + T^+}{2} + \left(\frac{-n^-}{2}\right)\left(T^- n_1 + T^+ n_2\right) = \frac{T^- + T^+}{2} - \frac{T^-}{2} + \frac{T^+}{2} = T^+$$

upstream

downstream

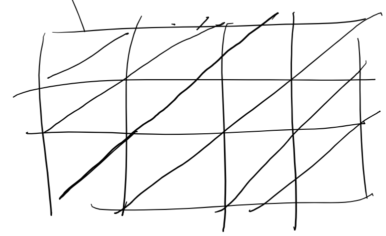
relative to V

$$q^* = q^- + \alpha[T]$$
$$T^* = T^+$$
$$q^* = q^+ + \alpha[T]$$
$$T^* = T^-$$



```
VECTOR *V_LDG, spatialNormal;
V_LDG = &phyConf->pdePseudoTimeMngr.V_LDG;
fldValsPtr->gPropPtr->facet_dForms[0].get_sdxSpaceNormalS(spatialNormal);
double Vn = Product(spatialNormal, *V_LDG);
int sgn = Sign(Vn);

if (sgn == -1)
{
    uAveuPart[e_IndexL] = false;         uAveuPart[e_IndexR] = true;
    uAveuFactor[e_IndexL] = 0.0;         uAveuFactor[e_IndexR] = 1.0;

    qAveqPart[e_IndexL] = true;              qAveqPart[e_IndexR] = false;
    qAveqFactor[e_IndexL] = 1.0;         qAveqFactor[e_IndexR] = 0.0;
}
else if (sgn == 1)
{
    uAveuPart[e_IndexL] = true;              uAveuPart[e_IndexR] = false;
    uAveuFactor[e_IndexL] = 1.0;         uAveuFactor[e_IndexR] = 0.0;

    qAveqPart[e_IndexL] = false;         qAveqPart[e_IndexR] = true;
    qAveqFactor[e_IndexL] = 0.0;         qAveqFactor[e_IndexR] = 1.0;
}
else
{
    uAveuPart[e_IndexL] = true;              uAveuPart[e_IndexR] = true;
    uAveuFactor[e_IndexL] = 0.5;         uAveuFactor[e_IndexR] = 0.5;

    qAveqPart[e_IndexL] = true;              qAveqPart[e_IndexR] = true;
    qAveqFactor[e_IndexL] = 0.5;         qAveqFactor[e_IndexR] = 0.5;
}
```

cannot choose these for V



Average $q^* = \frac{q^- + q^+}{2} + \alpha[T]$

$T^* = \frac{T^- + T^+}{2}$

which is better?

Alternating flux

$q^* = q^{\text{upstream}} + \alpha[T]$

$T^* = T^{\text{downstream}}$

better

DIFFERENT FORMULATIONS OF THE DISCONTINUOUS GALERKIN METHOD FOR THE VISCOUS TERMS*

CHI-WANG SHU†

Abstract. Discontinuous Galerkin method is a finite element method using completely discontinuous piecewise polynomial space for the numerical solution and the test functions. Until recently it was mainly used for solving convection problems involving only first spatial derivatives. Recently the method has been extended successfully to solve convection diffusion problems involving second derivative viscous terms. In this paper we will use simple examples to illustrate the basic ideas and "pitfalls" for using the discontinuous Galerkin method on the viscous terms.

Shu_2001_Different formulations of the discontinuous Galerkin method for the viscous terms.pdf

4. The local discontinuous Galerkin method for the second order diffusion problem. If we rewrite the heat equation (3.1) as a first order system

$$u_t - q_x = 0, \qquad q - u_x = 0, \tag{4.1}$$

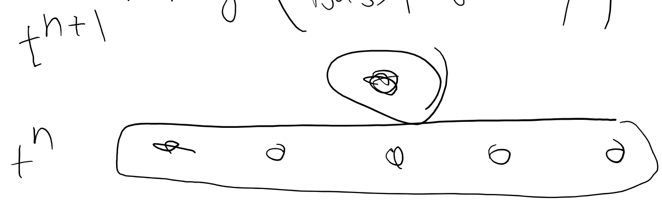
$$\hat{u}_{j+\frac{1}{2}} = \frac{1}{2}\left(u^-_{j+\frac{1}{2}} + u^+_{j+\frac{1}{2}}\right), \qquad \hat{q}_{j+\frac{1}{2}} = \frac{1}{2}\left(q^-_{j+\frac{1}{2}} + q^+_{j+\frac{1}{2}}\right). \tag{4.3}$$

$\hat{q}_{j+\frac{1}{2}}$ which guarantee stability, convergence and a sub-optimal error estimate of order k for piecewise polynomials of degree k. It turns out that the central fluxes (4.3)

2. The order of accuracy is one order lower for odd k. That is, for odd k the proof of the sub-optimal error estimate of order k is actually sharp.

$$\frac{d}{dt}u_j + \frac{1}{\Delta x_j^2}\left(Au_{j-2} + Bu_{j-1} + Cu_j + Du_{j+1} + Eu_{j+2}\right) = 0$$

Update stencil with average (Bassi-Rebay)



How about LDG alternating fluxes

2. The order of accuracy is one order lower for odd k. That is, for odd k the proof of the sub-optimal error estimate of order k is actually sharp.

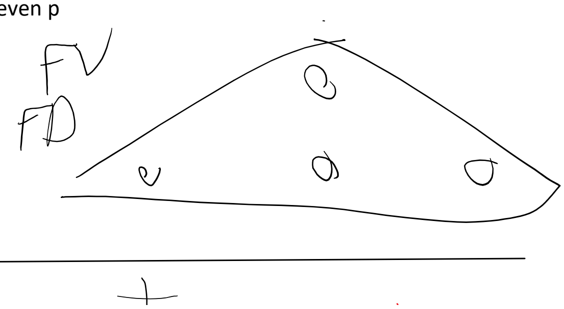
Both problems can be cured by a clever choice of fluxes, proposed in Cockburn and Shu [8]:

$$\hat{u}_{j+\frac{1}{2}} = u^-_{j+\frac{1}{2}}, \qquad \hat{q}_{j+\frac{1}{2}} = q^+_{j+\frac{1}{2}}. \tag{4.4}$$

Point 1: order of convergence is optimal (p + 1 for interpolation p) for both T and q for all off and even p

Point 2: stencil is much narrower:)

$$\frac{d}{dt}u_j + \frac{1}{\Delta x_j^2}\left(\cancel{Au_{j-2}} + Bu_{j-1} + Cu_j + Du_{j+1} + \cancel{Eu_{j+2}}\right) = 0$$

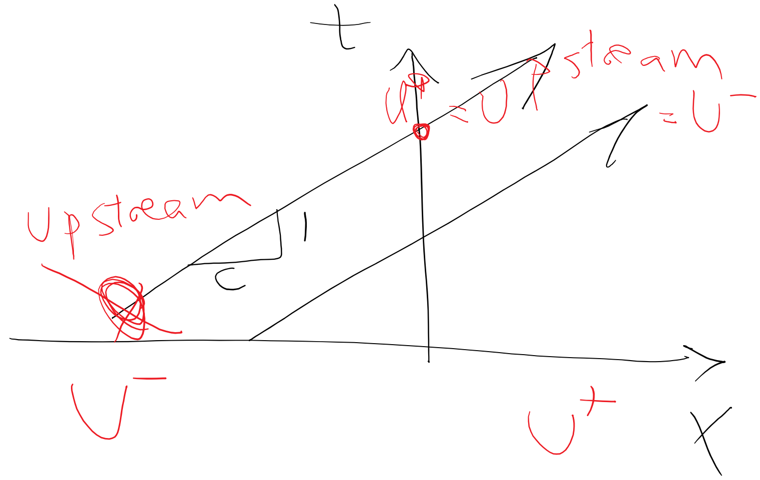


Last point about alternating fluxes

Last point about alternating fluxes

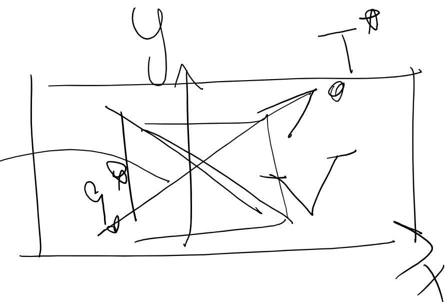
$$u_{,t} + c u_{,x} = 0$$

hyperbolic PDE



motivation for LDG / Alternating flux

"artificial wave direction" gives q^+, opposite direction: v^-



- Parabolic PDEs: will discuss fluxes later (after HW assignment), basically for parabolic PDEs we don't need alpha term for st ability as long as we don't want to get steady state solution.
- Also there is a way to find more physical fluxes (see Lorcher paper).
- Comparison of different DG fluxes and an erroneous flux option discussed in DG_course\Papers\Fluxes\Elliptic+Parabolic\ErroneousParabolicFluxes
  Read
  Shu_2001_Different formulations of the discontinuous Galerkin method for the viscous terms
  Kirby_2005_Selecting the Numerical Flux in Discontinuous Galerkin methods for diffusion problems

Hyperbolic PDEs

Solving this wave equation:

$$(1) \quad \boxed{c\ddot{u} + d\dot{u} - \underbrace{\nabla\cdot(K\nabla u)}_{\text{or another elliptic operator}} = S}$$

u : scalar

u : scalar
wave eqn

Elastodynamics $\quad u$ vector $\quad c \to \rho$

$d \to$ damping

$\nabla \cdot \cancel{k} \nabla u \to \nabla \cdot \sigma, \ \sigma = \underset{\text{stiffness}}{\overset{\equiv}{C}} \ \underset{\text{strain}}{\tilde{\nabla} u}$

Write (1) in the form of system of conservation laws:

$$\boxed{\begin{aligned} v &= \dot{u} \\ q &= \nabla u \end{aligned}}$$

velocity

"strain" like quantity

$$\dot{U} + \underbrace{F_{x}(U)_{,x} + F_{y}(U)_{,y} + F_{z}(U)_{,z}}_{\text{spatial flux in } x,y,z} = S$$

$$F = [F_x \mid F_y \mid F_z]$$

$$\dot{U} + \nabla \cdot F(U) = S$$

For semi-linear PDEs $\quad F$'s are linear functions of U

$$F_x(U) = A_x(\vec{x}, t)\, U \quad F_y(U) = A_y(\vec{x}, t)\, U \quad F_z(U) = A_z\, U$$

write $\quad c\ddot{u} + d\dot{u} - \nabla \cdot \cancel{k} \nabla u = S \quad$ in 1st PDE form:

$\dot{u} = v, \quad \nabla u = q$

source term

$U = V$, $\nabla u = q$

source term

$$\boxed{\begin{aligned} \dot U &= V && 2i \\ C\dot V - \nabla\cdot k q &= -dV && 2ii \text{ eqn } (1) \\ \dot q - \nabla V &= 0 && 2iii \end{aligned}}$$

$V = \dot U$

$q = \nabla u$

(2) $\dot q = \nabla \dot u = \nabla V$

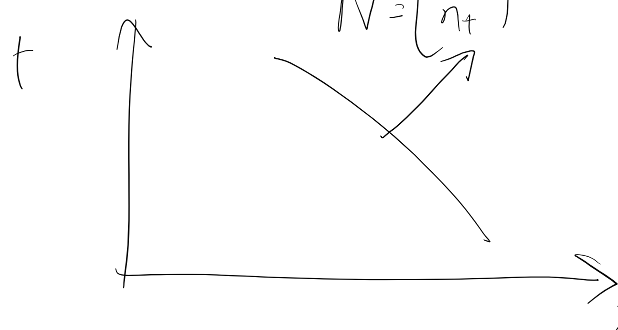
DG methods & jump conditions

$$\dot U + \nabla\cdot F(U) = f \qquad M = \begin{bmatrix} F(U) \\ U \end{bmatrix} \begin{matrix} \to \text{spatial} \\ \to \text{temporal} \end{matrix}$$

space-time flux

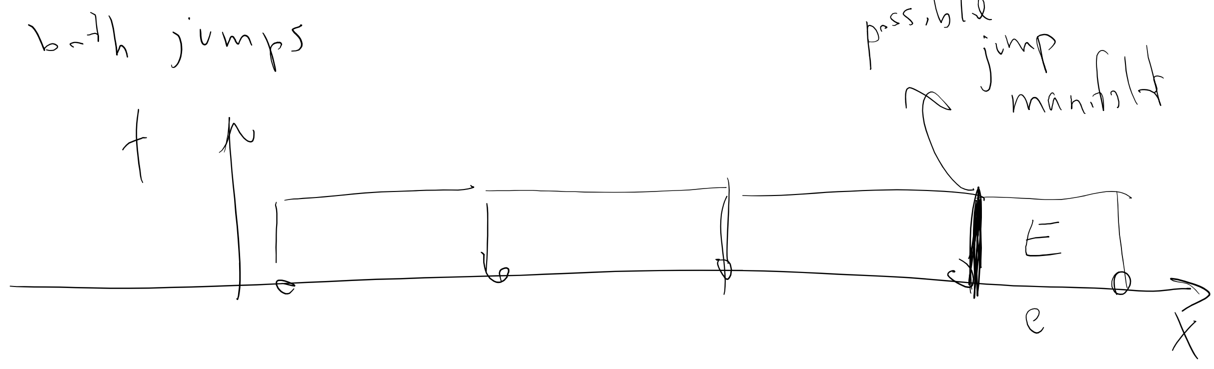
$$\vec N = \begin{bmatrix} n_x \\ n_t \end{bmatrix}$$

$$[M]\cdot N = 0$$



$$\boxed{[F(U)]\cdot n_x + [U]\cdot n_t = 0} \quad (3)$$

in DG methods
we need both jumps



$n_t = 0$

$$[F(U)]\, n_x = 0$$

need to model jump in $[F(U)]$ only between the elements