DG2020/02/26

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temporal spatial source

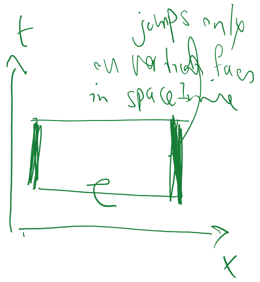
X $\hat{u} \leftarrow u \leftarrow \dot{u} = v$

X $\hat{v} \leftarrow v \leftarrow C\dot{v} + \nabla\cdot(-\kappa q) = -dv + S$

$\hat{q} \leftarrow q \leftarrow \dot{q} + \nabla(-v) = 0$

$\nabla\cdot 1 v$

Spatial fluxes



Diffusive/PDE part — **Jump term**

$\sigma = \kappa q$

$$\int_e \hat{u}(\dot{u}-v)\,dv +$$

$$\int_e \hat{v}\left(C\dot{v}+\nabla\cdot(-\kappa q)+dv-S\right)dv + \int_{\partial e}\hat{v}\left(-\sigma^* + \sigma\right)\cdot n\,ds$$

$$+\int_e \hat{q}\,(q-\nabla v)\,dv + \int_{\partial e}\hat{q}\cdot n\,(-v^* + v) = 0 \quad (1)$$

3 field formulation for wave eqn, DG space, time-marching in time

Since u does not have spatial flux, we can solve v and q from the last two lines of equations, then go back to the first lineand integrate v in time to get u.

$\int \hat{u}(\dot{u}-v)dv$ can be removed.

& u be solved in post-processing

Path to a single field formulation:

For a single-field formulation we strongly satisfy

$$\begin{cases} \dot{u} = v \\ \dot{q} = \nabla v \quad (q = \nabla u) \end{cases}$$

we solve only for u.

force-like fields

stress $\sim \sigma = \kappa \nabla u = \kappa q$

linear momentum density $\sim p = C\dot{u} = Cv$ (cond eqn)

Diffusive/PDE part — **Jump term**

$\sigma = \kappa q$

$$\cancel{\int_e \hat{u}(\dot{u}-v)\,dv} +$$

$$\int_e \hat{v}\left(C\dot{v}+\nabla\cdot(-\kappa q)+dv-S\right)dv + \int_{\partial e}\hat{v}\left(-\sigma_n^* + \sigma_n\right)ds.$$

$$+\cancel{\int_e \hat{q}\,(q-\nabla v)\,dv} + \int_{\partial e}\hat{q}\cdot n\,(-v^* + v) = 0 \quad (1)$$

$$\int_e \hat{\sigma}(\sigma - \kappa q)dv + \int_e \hat{p}(p - Cv)dv$$

added to WRS for 5 field formulation

Single-field formulation is

$$\int \underset{\hat{u}}{\hat{v}}\left(C\underset{\ddot{u}}{\dot{v}} + d\underset{\dot{u}}{v} - S - \nabla\cdot\underset{\kappa\nabla u}{\sigma}\right)dv + \int_{\partial e}\underset{\hat{u}}{\hat{v}}\left(-\sigma_n^* + \sigma_n\right)ds \quad (2)$$

$$\int_{\partial e}\hat{q}\cdot n\,(-v^* + v)\,ds = 0$$

WRS for single field formulation when only u is interpolated

n dof of element

$$u = \sum_{i=1}^{n} u_i(x)\, \underbrace{a_i(t)}_{\text{unknowns are function of time}}$$

$$q = \nabla u = \sum \nabla \hat{u}_i(x)\, a_i(t)$$

$$\hat{u}_i = u_i(x)$$

$$\hat{q}_i = \nabla \hat{u}_i = \nabla u_i$$

$$v(x,t) = \dot{u}(x,t) = \sum u_i(x)\, \dot{a}_i(t)$$

$$\hat{v}(x) = ? \quad \hat{u}_i(x) = u_i(x)$$

$$\hat{v} = \underbrace{[\hat{\dot{a}}_1(t) \; \dots \; \hat{\dot{a}}_n(t)]}_{\text{terms of weight}} \begin{bmatrix} u_1(x) \\ u_2(x) \\ \vdots \\ \hat{u}_n(x) \end{bmatrix} \qquad \hat{q} = [\hat{a}_1(t) \; \dots \; \hat{a}_n(t)] \begin{bmatrix} \nabla u_1 \\ \nabla u_2 \\ \vdots \\ \nabla u_n \end{bmatrix}$$

$$\dot{v} = [u_1 \; \dots \; u_n] \begin{bmatrix} \dot{a}_1 \\ \dot{a}_2 \\ \vdots \\ \dot{a}_n \end{bmatrix} \qquad q = [\nabla u_1 \; \dots \; \nabla u_n] \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

sh. unknowns

$$\hat{v} \to \hat{u}$$

$$\int \hat{u}\,(C\ddot{u} + d\dot{u} - \nabla\sigma - s)\,dv + \int_{\partial e} \hat{u}\,(-\hat{\sigma}_n + \sigma \cdot n)\,ds + \lambda' \int_{\partial e} \hat{q} \cdot n\,(-\hat{v} + v)\,ds = 0 \qquad (3)$$

one of these

λ' is for dimensional consistency

$$u = [u_1(x) \; \dots \; u_n(x)] \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \qquad q = [\nabla u_1 \; \dots \; \nabla u_n] \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

$$\hat{u} = \underbrace{[\hat{a}_1 \; \dots \; \hat{a}_n]}_{\text{arbitrary}} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad \hat{q} = [\hat{a}_1 \; \dots \; \hat{a}_n] \begin{bmatrix} \nabla u_1 \\ \vdots \\ \nabla u_n \end{bmatrix}$$

$$\cancel{[\hat{a}_1 \; \dots \; \hat{a}_n]} \left(\int \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} (C[u_1 \dots u_n])\,dv \begin{bmatrix} \ddot{a}_1 \\ \vdots \\ \ddot{a}_n \end{bmatrix} + \int \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} d\,[u_1 \; \dots \; u_n] \right) \begin{bmatrix} \dot{a}_1 \\ \vdots \\ \dot{a}_n \end{bmatrix}$$

$$+ \; \dots \; + \int \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} (-\hat{\sigma}_n)\,ds \; \dots \Big) = 0$$

Easier way $\hat{U} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ \vdots \\ U_n \end{bmatrix}$

$$\int \hat{U}\,(C\ddot{u} + d\dot{u} - \nabla\sigma - S)\,dv + \int_{\partial e} \hat{u}\,(-\sigma_n^{\star} + \sigma\cdot n)\,ds + \lambda \int_{\partial e} \hat{q}\cdot n\,(-V^{\star} + V)\,ds = 0 \qquad (4)$$

$\uparrow \hat{\sigma}$ one of these

$\hat{q} \sim \nabla \hat{u}$

q, $\sigma = K\,\nabla u$

For multi-field formulation terms multiplied by $\hat{U}, \hat{V}, \hat{q}$ are in different rows of stiffness matrix so there are not added together (no issue with dimensional consistency)

$[\lambda\, q\, V] = [u\, \sigma]$

$[\lambda]\,\cancel{[q]}\,[V] = [u]\,[K]\,\cancel{[q]}$

$\frac{[u]}{[V]} = T \to$ time

$[\lambda] = T\,[K]$

if we had used $\hat{\sigma}$ (★)

$[\lambda\, \sigma\, V] = [u\, \sigma]$

$[\lambda]\,\cancel{[\sigma]}\,[V] = [u]\,\cancel{[\sigma]}$

$[\lambda] = \frac{[u]}{[V]} = T \to$ time

element size h

Lambda must have the dimension of time. We can create a time scale from element size and wave speed

$$c = \sqrt{\frac{K}{C}} \qquad \lambda = \frac{h}{c} \qquad (5)$$

(4) & (5) (8) IBP

$$\int \hat{u}\,(-\nabla\cdot\sigma)\,dv + \int_{\partial e} \hat{u}\cdot\sigma n\,ds$$

$$\int_e [-\nabla\cdot\hat{u}\,\sigma + \nabla\hat{u}\,\sigma]\,dv + \int_{\partial e} \hat{u}\cdot\sigma n\,ds$$

$$= \cancel{\int_{\partial e} -\hat{u}\cdot\sigma n\,dv} + \boxed{\int_e \nabla\hat{u}\,\sigma\,dv} + \cancel{\int_{\partial e} \hat{u}\cdot\sigma n\,ds}$$

$$\int \hat{U}\,(C\ddot{u} + d\dot{u} - \nabla\sigma - S)\,dv + \int_{\partial e} \hat{u}\,(-\sigma_n^{\star} + \sigma\cdot n)\,ds + \lambda \int_{\partial e} \hat{q}\cdot n\,(-V^{\star} + V)\,ds = 0$$

$\uparrow \hat{\sigma}$ one of these

$$+\frac{\lambda}{\quad}\int_{\partial e} \hat{q}\cdot n\,(-V^{*}+V)\,ds = 0$$

(6)

$$\boxed{\begin{gathered}\int_e \left[\hat{u}(C\ddot{u} + d\dot{u} - S) + \nabla\hat{u}\,\sigma\right] dv + \int_{\partial e} \hat{u}(-\sigma^{*}\cdot n)\,ds \\ \sigma = k\nabla u \\ + \lambda\int \hat{\sigma}\cdot n(-V^{*}+V)\,ds = 0\end{gathered}}$$

$\lambda = 1$ (non dimensionally consistent)

$\lambda = \frac{h}{c}$ — "

$$u = \sum a_i(x)\,a_i(t) \qquad \ddot{u} = [u_1(x) \cdots u_n(x)]\begin{bmatrix} \ddot{a}_1(t) \\ \ddot{a}_2(t) \\ \vdots \\ \ddot{a}_n(t)\end{bmatrix} \qquad \dot{u}(x,t) = [u_1(x) \cdots u_n(x)]\begin{bmatrix} a_1(t) \\ a_2(t) \\ a(t) \\ \vdots \\ \dot{a}_n(t)\end{bmatrix}$$

$$\ddot{u}(x,t) = \underbrace{[u_1(x) \cdots u_n(x)]}_{U(x)}\begin{bmatrix} \ddot{a}_1(t) \\ \vdots \\ \ddot{a}_n(t)\end{bmatrix} \qquad \nabla u = [\nabla u_1 \cdots \nabla u_n(x)]\begin{bmatrix} a_1 \\ \vdots \\ a_n\end{bmatrix}$$

$$\hat{u} = \begin{bmatrix} u_1(x) \\ \vdots \\ u_n(x)\end{bmatrix} \qquad \nabla\hat{u} = \begin{bmatrix} \nabla u_1 \\ \vdots \\ \nabla u_n\end{bmatrix} \qquad \hat{\sigma} = k\begin{bmatrix} \nabla u_1 \\ \vdots \\ \nabla a_n\end{bmatrix}$$

plug into the equation

(7)

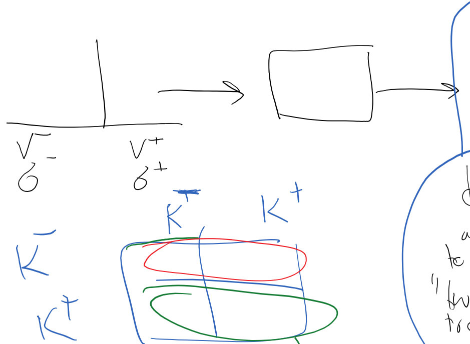
$$\boxed{\begin{gathered}\underbrace{\left(\int_e U^T C U\,dv\right)}_{M^e}\ddot{a} + \underbrace{\left(\int U^T d U\,dv\right)}_{C_b^e}\dot{a} + \underbrace{\left(\int_e \nabla U^T k \nabla U\,dv\right)}_{K_b^e \text{ (bulk)}} a \\ - \underbrace{\int_e U^T S\,dv}_{F_r^e} + \int_{\partial e}\hat{U}(-\sigma_n^{*})\,ds + \lambda\int \hat{\sigma}\cdot n(-V^{*}+V)\,ds = 0 \\ M^e\ddot{a} + C_b^e\dot{a} + K_b^e a - F_r^e + \int_{\partial e}\hat{u}(-\sigma_n^{*})\,ds + \lambda\int \hat{\sigma}\cdot n(-V^{*}+V)\,ds = 0\end{gathered}}$$

1-field formulation semi-descrete statement.

Next step is calculating star values

$$R = M\ddot{a} + C\dot{a} + Ka - F \qquad M\ddot{a} + C\dot{a} + Ka = F$$

We need to write a black-box type function that computes start values AND their derivatives



values $\begin{bmatrix} V^{*}(V^-, V^+, \sigma^-, \sigma^+) \\ \sigma^{*}(V^-, V^+, \sigma^-, \sigma^+)\end{bmatrix}$

derivatives with respect to arguments "two side traces"

$$\begin{bmatrix} \frac{\partial V^*}{\partial V^-}, & \frac{\partial V^*}{\partial V^+}, & \frac{\partial V^*}{\partial \sigma^-}, & \frac{\partial V^*}{\partial \sigma^+} \\ \frac{\partial \sigma^*}{\partial V^-} & \frac{\partial \sigma^*}{\partial V^+} & \frac{\partial \sigma^*}{\partial \sigma^-} & \frac{\partial \sigma^*}{\partial \sigma^+}\end{bmatrix}$$