DG2020/03/02

Monday, March 2, 2020 11:39 AM

From last time:

$$\overbrace{\left(\int_e U^T C U\, dv\right)}^{m^e} \ddot{a} + \underbrace{\left(\int_e U^T d U\, dv\right)}_{C_b^e} \dot{a} + \left[\int_e (\nabla U^T \kappa \nabla U)\, dv\right] a - \int_e U^T s\, dv + \int_{\partial e} \hat{U}(-\sigma_n^{*})\, ds + \int_{\partial e} \hat{\sigma}\cdot n(-v^{*}+v)\, ds = 0 \qquad (1)$$

Deriving the matrices for linear elements in 1D

1D, p = 1

$x = \frac{X - x_0}{h}$, $U = [1 \;\; x]$, $\nabla U = \frac{dU}{dX} = [0 \;\; \frac{1}{h}]$

$$m^e = \int_e U^T C U\, dv = \int_0^1 \begin{bmatrix}1\\ x\end{bmatrix} c\,[1 \;\; x]\,(h\, dx) = ch\begin{bmatrix}1 & 1/2\\ 1/2 & 1/3\end{bmatrix}$$

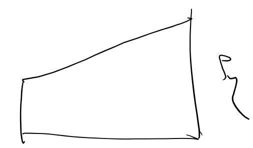
$$C_b^e = \int U^T d U\, dv = dh\, \cdots$$

Damping matrix from the interior (bulk) of the element

$$K_b^e = \int \nabla U^T \kappa \nabla U = \int_0^1 \begin{bmatrix}0\\ 1/h\end{bmatrix} \kappa\, [0 \;\; 1/h]\,(h\, dx) = \frac{\kappa}{h}\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$$

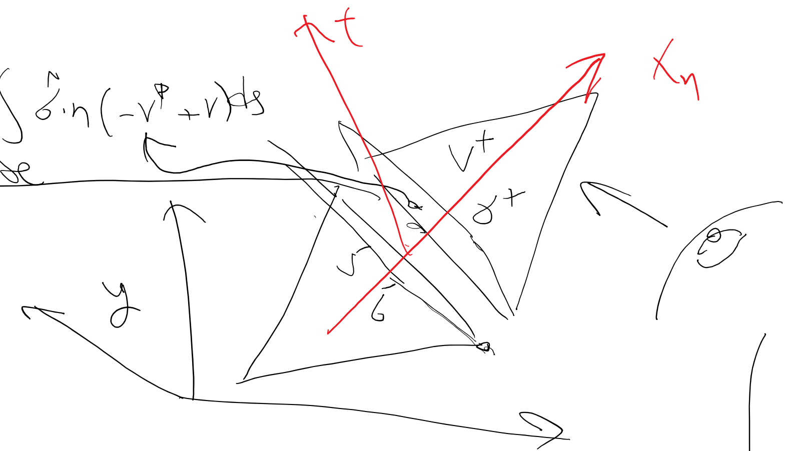
Force term

$$\int U^T s\, dv = Q\begin{bmatrix}s_1\\ s_2\end{bmatrix} \qquad Q = h\begin{bmatrix}1/2 & 1/2\\ 1/6 & 1/3\end{bmatrix}$$



Now we focus on terms related to boundary jumps:

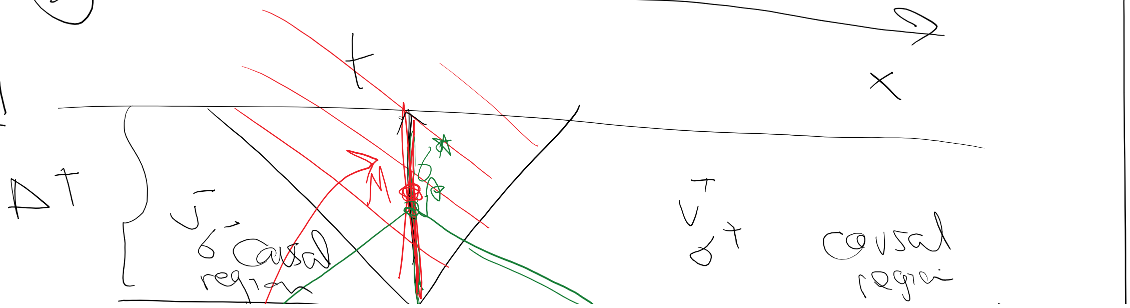
$$\int_{\partial e} \hat{U}(-\sigma_n^{*})\, ds + \int_{\partial e} \hat{\sigma}\cdot n(-v^{*}+v)\, ds$$

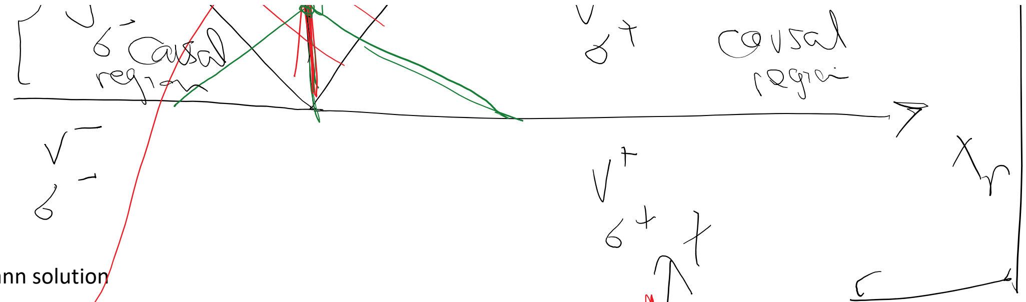


Choices are average fluxes

$$v_A^* = \frac{v^- + v^+}{2} \qquad \sigma_A^* = \frac{\sigma^- + \sigma^+}{2} \qquad (2)$$

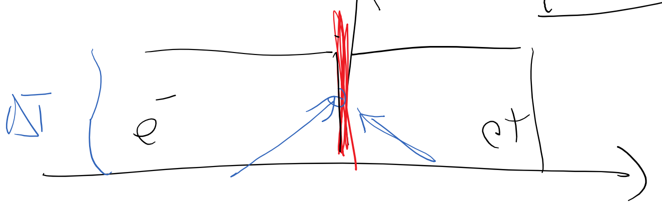
Or use Riemann solutions





We find the solution on vertical line by a Riemann solution

With time marching schemes we only care about (approximate) Riemann solution on vertical lines.
If we have source terms, we need to calculate Riemann solution form time 0 to Delta t and average it over the red line



or computing the value at mid-point M

Solution procedure

$\dot{v} - \frac{1}{\rho}\sigma_{,x} = 0$ (balance law)

$\dot{\sigma} - E v_{,x} = 0$ (compatibility)

$\dot{\varepsilon} = \tilde{\nabla} v$

$\dot{q} + A q_{,x} = 0 \qquad A = \begin{bmatrix} 0 & -\frac{1}{\rho} \\ -E & 0 \end{bmatrix}$

left eigenvalues $[l_1 \cdots l_n] A = \lambda [l_1 \cdots l_n]$

eigen decomposition

$$\underbrace{\begin{bmatrix} l^{(1)} \\ l^{(2)} \end{bmatrix}}_{L} A = \begin{bmatrix} \lambda^1 & \\ & \lambda^2 \end{bmatrix} \begin{bmatrix} l^{(1)} \\ l^{(2)} \end{bmatrix} \qquad \boxed{LA = \Lambda L}$$

$\dot{q} + A q_{,x} = 0$ premultiply by L

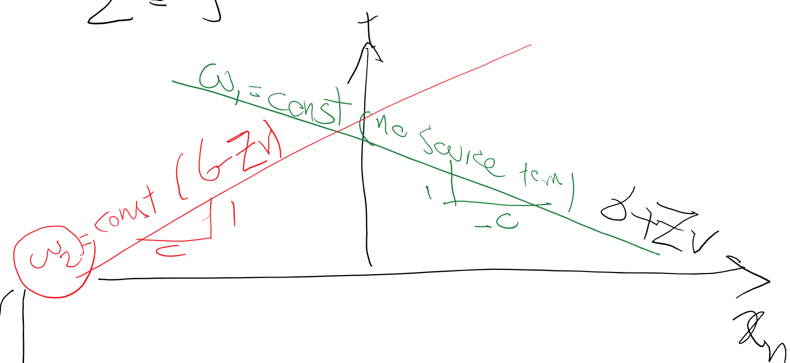
$$(L\dot{q}) + \underbrace{LA}_{\Lambda L} q_{,x} = 0 \qquad \underbrace{(Lq)}_{\text{characteristic parameter } \omega}\dot{} + \Lambda (Lq)_{,x} = 0$$

$$\boxed{\begin{aligned} &\omega = Lq \\ &\dot{\omega} + \Lambda \omega_{,x} = 0 \end{aligned}}$$

L for $A = \begin{bmatrix} 0 & -\frac{1}{\rho} \\ -E & 0 \end{bmatrix}$ is $L = \begin{bmatrix} c\rho & 1 \\ -c\rho & 1 \end{bmatrix}$ & $\Lambda = \begin{bmatrix} -c & 0 \\ 0 & c \end{bmatrix}$

$c = \sqrt{E/\rho} \quad (\rho\ddot{u} - E u_{,xx}) \qquad Z = c\rho$ impedance

$$\boxed{\begin{aligned} &\dot{\omega}_1 + \lambda_1 \omega_{1,x} = 0 \qquad \lambda_1 = -c \\ &\dot{\omega}_2 + \lambda_2 \omega_{2,x} = 0 \qquad \lambda_2 = c \end{aligned}}$$



$$\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} Z & 1 \\ -Z & 1 \end{bmatrix} \begin{bmatrix} v \\ \sigma \end{bmatrix} = \begin{bmatrix} \sigma + Zv \\ \sigma - Zv \end{bmatrix}$$

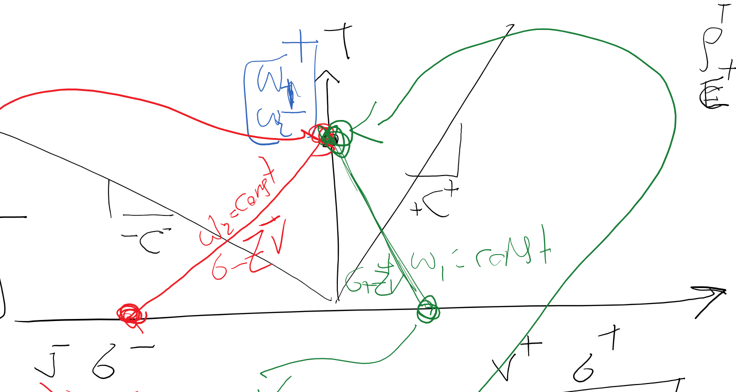
the Riemann sln
by using characteristics we can solve

by using

$$\bar{c}^- = \sqrt{\frac{E^-}{\rho^-}} \qquad \begin{cases} \rho^- \\ E^- \end{cases}$$

$$Z^- = \sqrt{E^-\rho^-}$$

$$\sigma^- - Z^- v^- = \sigma^* - Z^- v^*$$



ρ^+, E^+, $c^+ = \sqrt{\frac{E^+}{\rho^+}}$

$$\sigma^+ + Z^+ v^+ = \sigma^* + Z^+ v^*$$

$$\left.\begin{aligned} \sigma^* - Z^- v^* &= \sigma^- - Z^- v^- \\ \sigma^* + Z^+ v^* &= \sigma^+ + Z^+ v^+ \end{aligned}\right\} \longrightarrow$$

$$\sigma^* = \left(\frac{Z^-\sigma^+ + Z^+\sigma^-}{Z^- + Z^+}\right) + \frac{Z^- Z^+}{Z^- + Z^+}\left(v^+ - v^-\right)$$

appear in Riemann sln

$$v^* = \frac{1}{Z^- + Z^+}\left(\sigma^+ - \sigma^-\right) + \frac{Z^- v^- + Z^+ v^+}{Z^- + Z^+}$$

if impedance are equal $Z^- = Z^+ = Z$

$$\sigma^* = \left(\frac{\sigma^- + \sigma^+}{2}\right) + \frac{Z}{2}\left(v^+ - v^-\right)$$

$$v^* = \left(\frac{v^- + v^+}{2}\right) + \frac{1}{2Z}\left(\sigma^+ - \sigma^-\right) \qquad (3)$$

average flux

some added jump terms

will add to damping matrix

General expression of * solutions for this linear PDE:

$$\sigma^* = \Sigma_{\sigma^-}\sigma^- + \Sigma_{\sigma^+}\sigma^+ + \Sigma_{v^-}v^- + \Sigma_{v^+}v^+$$

$$v^* = V_{\sigma^-}\sigma^- + V_{\sigma^+}\sigma^+ + V_{v^-}v^- + V_{v^+}v^+$$

$$\frac{\partial\sigma^*}{\partial[\sigma^-, \sigma^+, v^-, v^+]} = \begin{bmatrix} \Sigma_{\sigma^-} \\ \Sigma_{\sigma^+} \\ \Sigma_{v^-} \\ \Sigma_{v^+} \end{bmatrix}$$

& similar for v^*

& similar for ν^{∂}

Expressions of these coefficients

(A) average

$$\begin{cases} \Sigma_{\sigma^-} = \frac{1}{2} & \Sigma_{\sigma^+} = \frac{1}{2} & \Sigma_{v^-} = 0 & \Sigma_{v^+} = 0 & \left(\sigma^* = \left(\frac{\sigma^- + \sigma^+}{2}\right)\right) \\ V_{\sigma^-} = 0 & V_{\sigma^+} = 0 & V_{v^-} = \frac{1}{2} & V_{v^+} = \frac{1}{2} & \left(v^* = \left(\frac{v^- + v^+}{2}\right)\right) \end{cases}$$

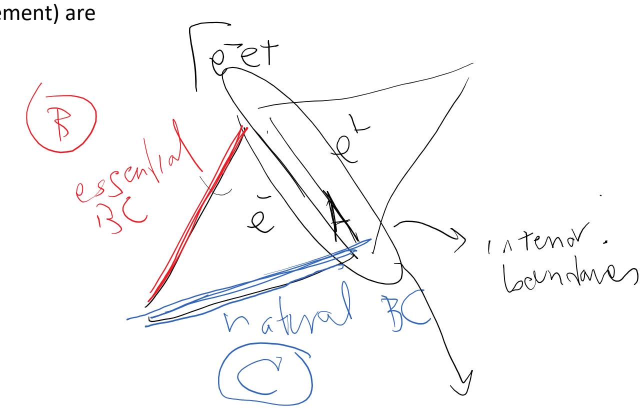
(R) Riemann

$$\begin{cases} \Sigma_{\sigma^-} = \frac{\bar{Z}^+}{\bar{Z}^- + \bar{Z}^+} & \Sigma_{\sigma^+} = \frac{\bar{Z}^-}{\bar{Z}^- + \bar{Z}^+} & \Sigma_{v^-} = \frac{-\bar{Z}^-\bar{Z}^+}{\bar{Z}^- + \bar{Z}^+} & \Sigma_{v^+} = -\Sigma_{v^-} \\ V_{\sigma^-} = \frac{-1}{\bar{Z}^- + \bar{Z}^+} & V_{\sigma^+} = -V_{\sigma^-} & V_{v^-} = \frac{\bar{Z}^-}{\bar{Z}^- + \bar{Z}^+} & V_{v^+} = \frac{\bar{Z}^+}{\bar{Z}^- + \bar{Z}^+} \end{cases} \quad (4)$$

We'll talk about star values for nonlinear PDEs later

So, the only two terms that we have not discussed in detail yet in equation (1) (weak statement) are

$$\underbrace{\int_{\partial e} \hat{v}\,(-\sigma_n^*)\,ds}_{I_U} + \underbrace{\lambda \int_{\Gamma_{e^-e^+}} \hat{\sigma}\cdot n\,(-v^* + v)\,ds}_{I_\sigma}$$



$\sigma_n^* = \sigma^* \cdot n$

$n = n_{\text{e}gn}$

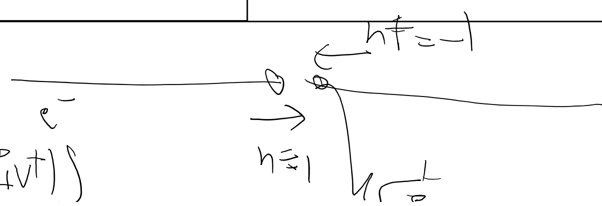
Case A: Interior of the domain contribution: We'll try to cast it in the form of IP method:

$$\int_{\Gamma_{e^-e^+}} \hat{U}^-(-\sigma^* \cdot n^-)\,ds + \lambda^- \int_{\Gamma_{e^-e^+}} \hat{\sigma}^- \cdot n^-\,(-v^* + v^-)\,ds$$

$$+ \int_{\Gamma_{e^-e^+}} \hat{U}^+(-\sigma^* \cdot n^+)\,ds + \lambda^+ \int_{\Gamma_{e^-e^+}} \hat{\sigma}^+ \cdot n^+\,(-v^* + v^+)\,ds$$

$$B_I(\hat{U}, U) \to B_I(\hat{U}, U) = \int_{\Gamma_{e^-e^+}} -\underbrace{(\hat{U}^- n^- + \hat{U}^+ n^+)}_{J_U}\sigma^*\,ds + \int_{\Gamma_{e^-e^+}} \underbrace{\left[\lambda^- \hat{\sigma}^- n^-(-v^* + v^-) + \lambda^+ \hat{\sigma}^+ \cdot n^+(-v^* + v^+)\right]}_{J_\sigma}\,ds \quad (5)$$

Simplify the equation above for 1D case:



$$B_{\Gamma_{e^-e^+}} = [\![\hat{U}]\!]\,\sigma^* + \left(\lambda^- \hat{\sigma} n^-(-v^* + v^-)\ \lambda^+ \hat{\sigma}^+ n^+(-v^* + v^+)\right) \quad (6)$$

$$B_{\Gamma_{e^- e^+}} = \underbrace{[\hat{U}]\,\sigma^*}_{J_u} + \underbrace{\left(\bar{\lambda}\bar{\hat{\sigma}} n^- (-\bar{V} + \hat{V}^+)\ \lambda^+ \hat{\sigma}^+ n^+ (-\hat{V} + V^+)\right)}_{J_\sigma} \quad n = 1 \quad \Big|_{\Gamma_{e^- e^+}} \qquad (6)$$

Before evaluating equation (6) let's simply fields from left and right

$$[\Phi] = \phi^- - \phi^+$$

$$U = [1 \quad x] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$\nabla u = u_{,x} = [0 \quad \tfrac{1}{h}] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$\sigma = k \nabla u = [0 \quad \tfrac{k}{h}] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$V = \dot{U} = \frac{d}{dt}\left([1 \quad x] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = [1 \quad x] \begin{bmatrix} \dot{a}_1 \\ \dot{a}_2 \end{bmatrix}$$

$$u^- = [1 \quad x^-] \begin{bmatrix} a_1^- \\ a_2^- \end{bmatrix} \quad \text{dof 1, dof 2} \qquad u^+ = [1 \quad x^+] \begin{bmatrix} a_1^+ \\ a_2^+ \end{bmatrix} \quad \text{dof 3, dof 4}$$

$$x^- = \frac{X - x_0^-}{h^-} \qquad x^- = 1 \qquad x^+ = 0$$

$$U^- = [1 \quad 1]\, a^- \qquad U^+ = [1 \quad 0]\, a^+$$

$$\sigma^- = \frac{k^-}{h^-}[0 \quad 1]\, a^- \qquad \sigma^+ = [0 \quad \tfrac{k^+}{h^+}]\, a^+$$

$$V^- = [1 \quad 1]\, \dot{a}^- \qquad V^+ = [1 \quad 0]\, \dot{a}^+$$

we need $[\hat{u}]$, $V^\mp$, $\sigma^\mp$, $\bar{\lambda}\bar{\hat{\sigma}}\bar{n}$, $\lambda^+\hat{\sigma}^+ n^+$

$$[U] = U^- n^- + U^+ n^+ = U^- - U^+ = \left([1 \quad 1 | 0 \quad 0] - [1 \quad 0 | 0 \quad 0]\right) a = [1 \quad 1 \quad -1 \quad 0]\, a$$

$$V^- = [1 \quad 1 \quad 0 \quad 0]\, \dot{a} \qquad V^+ = [0 \quad 0 \quad 1 \quad 0]\, \dot{a}$$

$$\lambda^- \hat{\sigma}^- n^- = [0 \quad \tfrac{k^- \lambda^-}{h^-} | 0 \quad 0]^T \quad (n^- = 1) \qquad \lambda^+ \hat{\sigma}^+ n^+ = [0 \quad 0 \quad 0 \quad -\tfrac{\lambda^+ k^+}{h^+}]^T \quad (n^+ = -1)$$

$$\lambda \hat{\sigma} n := \lambda^- \hat{\sigma}^- n^- + \lambda^+ \hat{\sigma}^+ n^+ = \left[0 \quad \frac{k^- \lambda^-}{h^-} \quad 0 \quad \frac{-\lambda^+ k^+}{h^+}\right] \qquad (7)$$

$$\sigma^- = [0 \quad \tfrac{k^-}{h^-} \quad 0 \quad 0]\, a \qquad \sigma^+ = [0 \quad 0 \quad 0 \quad \tfrac{k^+}{h^+}]\, a$$

Terms for stiffness

$$\sigma^* = \Sigma_{\sigma^-}\sigma^- + \Sigma_{\sigma^+}\sigma^+ + \Sigma_{V^-} V^- + \Sigma_{V^+} V^+$$

$$J_U = -[U]\,\sigma^* = -\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}\left(\Sigma_{\sigma^-}\sigma^- + \Sigma_{\sigma^+}\sigma^+ + \Sigma_{V^-} V^- + \Sigma_{V^+} V^+\right)$$

$$= \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}\Big(\Sigma_{\sigma^-}[0 \quad \tfrac{k^-}{h^-} \| 0 \quad 0]\, a + \Sigma_{\sigma^+}[0 \quad 0 \| 0 \quad \tfrac{k^+}{h^+}]\, a$$

$$+ \Sigma_{V^-}[1 \quad 1 \| 0 \quad 0]\, \dot{a} + \Sigma_{V^+}[0 \quad 0 | 1 \quad 0]\, \dot{a}\Big)$$

$\neq 0$ for Riemann & these terms will be added

$\ne 0$ for Riemann & these terms will be added to damping

$$J_U = -\begin{bmatrix} \vdots \\ - \\ 0 \end{bmatrix} \left(\Sigma_{\sigma^-} \begin{bmatrix} 0 & \frac{k^-}{h^-} & 0 & 0 \end{bmatrix} + \Sigma_{\sigma^+} \begin{bmatrix} 0 & 0 & 0 & \frac{k^+}{h^+} \end{bmatrix} \right)$$

goes to $K^I_{4\times4}$

$$- \begin{bmatrix} \vdots \\ - \\ 0 \end{bmatrix} \left(\Sigma_{v^-} \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} + \Sigma_{v^+} \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \right)$$

goes to $C^I_{4\times4}$
interface damping