DG2020/03/04

Wednesday, March 4, 2020 11:40 AM

So, last time we worked on the expression of this term:

$$\hat{I}_{\hat{u}} = -[\hat{u}]\,\sigma^*$$

This time, we'll work on the expression of $\tilde{I}_\sigma$

$$\tilde{I}_\sigma = \lambda^- \hat{\sigma}^- . n^- (-v^* + v^-) + \lambda^+ \hat{\sigma}^+ . n^+ (-v^* + v^+)$$

$n^+ = 1$

$n^- = -1$

$$V^* = V_{\sigma^-}\sigma^- + V_{\sigma^+}\sigma^+ + V_{v^-}v^- + V_{v^+}v^+$$

$$\rightarrow \tilde{I}_\sigma = (\lambda\hat{\sigma}n)(-v^*) + \lambda^-\hat{\sigma}^-.n^- v^- + \lambda^+\hat{\sigma}^+ n^+ v^+$$

one word

$$\lambda\hat{\sigma}n = \lambda^-\hat{\sigma}^- n^- + \lambda^+\hat{\sigma}^+ n^+$$

$$= -(\lambda\hat{\sigma}n)[\underbrace{V_{\sigma^-}\sigma^- + V_{\sigma^+}\sigma^+}_{\rightarrow K_2} + \underbrace{V_{v^-}v^- + V_{v^+}v^+}_{\rightarrow C_2}]$$

$$\underbrace{+ (\lambda\hat{\sigma}n)^- v^- + (\lambda\hat{\sigma}n)^+ v^+}_{\rightarrow C_2} = K_2 a + C_2 \dot{a}$$

(K_1, C_1 are from $I_{\hat{u}}$)

$$K_2 = -(\lambda\hat{\sigma}n)\left(V_{\sigma^-}\underbrace{\begin{bmatrix} 0 \\ K^-/h^- \\ 0 \\ 0 \end{bmatrix}^T}_{\sigma^-} + V_{\sigma^+}\underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ K^+/h^+ \end{bmatrix}^T}_{\sigma^+}\right)$$

$$C_2 = -(\lambda\hat{\sigma}n)\left(V_{v^-}\underbrace{[1 \;\; 1 \;|\; 0 \;\; 0]}_{v^-} + V_{v^+}\underbrace{[0 \;\; 0 \;|\; 1 \;\; 0]}_{v^+}\right)$$

$$+ (\lambda\hat{\sigma}n)^- \underbrace{[1 \;\; 1 \;\; 0 \;\; 0]}_{v^-} + (\lambda\hat{\sigma}n)^+ \underbrace{[0 \;\; 0 \;\; 1 \;\; 0]}_{v^+}$$

$$(\lambda\hat{\sigma}n)^- = \begin{bmatrix} 0 \\ -\lambda^- \cancel{K^-}/h^- \\ 0 \\ 0 \end{bmatrix} \quad (\lambda\hat{\sigma}n)^+ = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\dfrac{\lambda^+ \cancel{K^+}}{h^+} \end{bmatrix} \quad \lambda\hat{\sigma}n = \begin{bmatrix} 0 \\ -\lambda^- \cancel{K^-}/h^- \\ 0 \\ -\dfrac{\lambda^+ \cancel{K^+}}{h^+} \end{bmatrix}$$

(1)

$$K^I = \overset{\text{last time}}{K_1} + K_2 \rightarrow \text{(1) this time} \qquad C^I = C_1 + C_2$$

we assemble them to the global system a before

dof: 1, 2 — e_1 ... 2m−1, 2m — e_m ... 2m+1, 2m+2 — e_{m+1}

$$C^I \quad \begin{array}{c} 2m-1 \\ \vdots \\ 2m+2 \end{array}\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \rightarrow \text{global } C \quad C$$

Σ same for K^I $\longrightarrow$ K

Mass matrix is block diagonal

$$M\ddot{a} + C\dot{a} + Ka = F$$

$$M\left(\frac{a_{n+1} + a_{n-1} - 2a_n}{\Delta t^2}\right) + C\left(\frac{a_{n+1} - a_n}{\Delta t}\right) + Ka_n = F_n$$

$$(M + C\Delta t)\, a_{n+1} = \ldots \quad RHS$$

$M \rightarrow$ block-diagonal

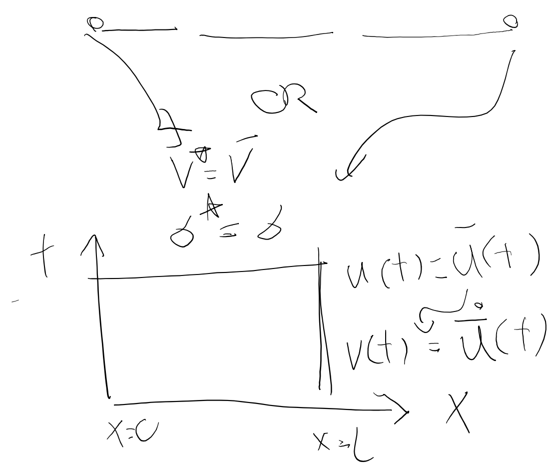
$C\Delta t \rightarrow$ C is not

If we wanted to solve a block-diagonal system (so it would enable one element solution as a time), a 2 or 3 field formulation for which u and v are independently interpolated would have been needed.

Essential boundary:

$$B(\hat{u}, u) = -\int_{\partial e} \hat{u}(\sigma \cdot n)\, ds + \lambda \int_{\partial e} \tilde{\sigma} \cdot n(-V^* + V)\, ds$$

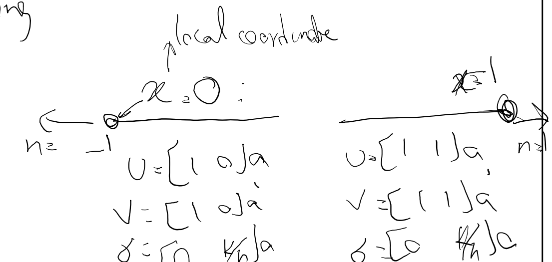
in 1D $\partial e \rightarrow$ point



$$B(\hat{u}, u) = \underbrace{-\hat{u}(\sigma \cdot n)}_{\text{stiffness}} + \lambda\, \hat{\sigma} \cdot n(-\bar{V} + V)$$

RHS damping

$V = \ldots \dot{a}$



$K_{\text{contribution}} = -\hat{u}(\sigma \cdot n)$

$\sigma = [0 \quad k/h]a \qquad \sigma = [0 \quad k/h]a$

$$K_{\text{contribution}} = -\hat{u}(\sigma \cdot n) \qquad \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}0 & k/h\end{bmatrix} \qquad -\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}0 & k/h\end{bmatrix}$$

$$C_{\text{contribution}} = \lambda \hat{\sigma} \cdot n v \qquad -\lambda\begin{bmatrix}0\\k/h\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} \qquad \lambda\begin{bmatrix}0\\k/h\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix}$$

$$F_{(\text{force})} = \lambda \hat{\sigma} \cdot n \bar{v} \qquad -\lambda\begin{bmatrix}0\\k/h\end{bmatrix}\bar{v} \qquad \lambda\begin{bmatrix}0\\k/h\end{bmatrix}\bar{v}$$

② Essential BC

Natural boundary

$$B(\hat{u},u) = -\int_{\partial \Omega} \hat{u}(\sigma_n^*)\,ds + \lambda \int \hat{\sigma} \cdot n\,(-v^* + v)\,ds$$

$v^* = v$

$\sigma_n^* = \bar{\sigma}$

1D

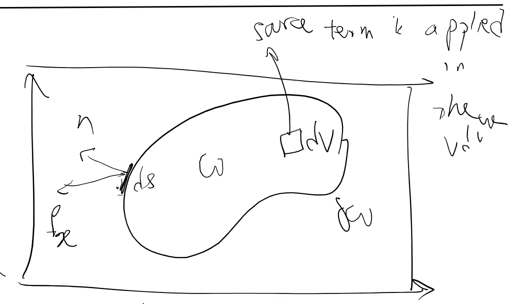
$$B(\hat{u},u) = \underbrace{-\hat{u} \cdot \bar{\sigma}}_{\text{RHS force}}$$

$$\hat{u} = \begin{bmatrix}1\\0\end{bmatrix} \qquad \hat{u} = \begin{bmatrix}1\\1\end{bmatrix}$$

$$F = \begin{bmatrix}1\\0\end{bmatrix}\bar{\sigma} \qquad F = \begin{bmatrix}1\\1\end{bmatrix}\bar{\sigma}$$

Steady state balance laws

$$-\underbrace{\int_{\partial \Omega} f_x \cdot n\,ds}_{\text{outward flux}} + \int_{\Omega} s\,dV = 0$$



net outward flux density

$f_x \cdot n$

Examples

solid Mechanics $\quad f_x = -\sigma \qquad s = \rho b \qquad \int_{\partial \Omega} \sigma \cdot n\,ds + \int_{\Omega} \rho b\,dv = 0$

Steady state heat conduction

$f_x = q \qquad s = Q \qquad \int_{\partial \Omega} q \cdot n\,ds + \int_{\Omega} Q\,dv = 0$

What if the problem is dynamic?

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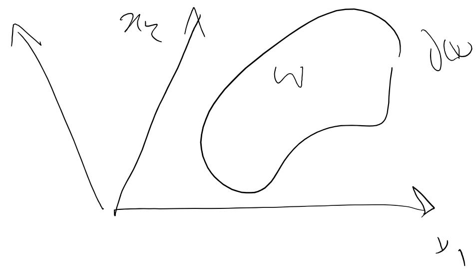
heat conduction

$$-\int_{\partial\omega} q\cdot n\,ds + \int_{\omega} Q\,dv = \frac{d}{dt}\mathcal{E} \qquad \text{balance of energy}$$



$$= \frac{d}{dt}\int e\,dv$$

e: energy density (per volume)

simple model (ignoring other contributions)

$$e(T) = C_v T$$

C_v: volumetric heat capacity

$$-\int q\cdot n\,ds + \int Q\,dv = \frac{d}{dt}\int (C_v T)\,dv$$

$$\boxed{\frac{d}{dt}\underbrace{\int_{\omega}(\overbrace{C_v T}^{\text{energy density}})\,dv}_{\text{energy}} + \underbrace{\int_{\partial\omega} q\cdot n\,ds}_{\text{spatial flux}} = \underbrace{\int_{\omega} Q\,dv}_{\text{source term}}}$$

balance of energy (3)

$$\frac{d}{dt}\int_{\omega}\underbrace{\rho\vec{v}}_{\text{lin. mom. density}}\,dv + \int_{\partial\omega}(-\sigma)\cdot n\,ds = \int \rho b\,dv \qquad \text{balance of linear momentum}$$

$\frac{dP}{dt}$

$$[\rho v]_{,t} + [-\sigma + \rho v v]_{,x} = 0 \quad \text{solid}$$

$$\frac{\partial}{\partial t}\int_{\omega}\rho\,dv + \int_{\partial\omega}\rho\vec{v}\cdot n\,ds = 0 \qquad \text{Mass}$$

Any dynamic balance law can be written as

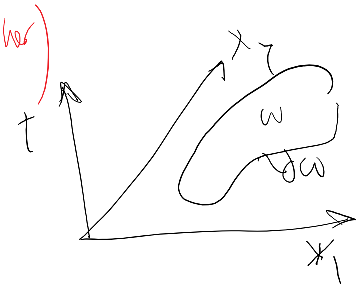
$$\frac{d}{dt}F = \frac{d}{dt}\int_{\omega} f_t \, dV = -\int_{\partial\omega} f_x \cdot n \, dS + \int_{\omega} S \, dV$$

Balanced quantity (energy, mass, ...)

temporal flux = density of balanced quantity

outward spatial flux density (1 tensor order higher than f_t)

source term



④

④ ⟶ go to a spacetime expression of balance law

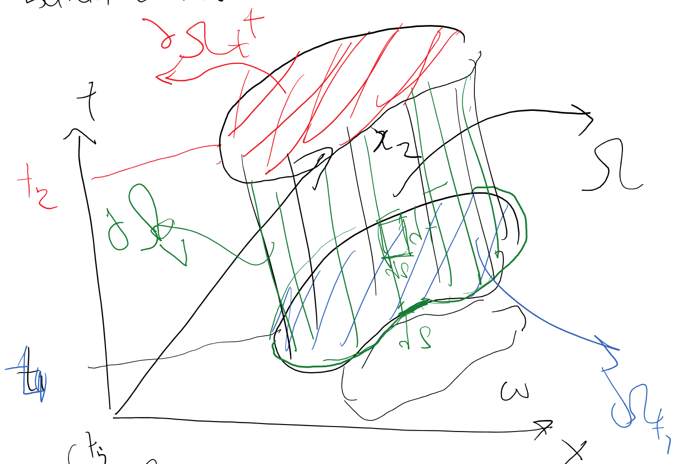
$$\frac{d}{dt}\int_{\omega} f_t \, dV + \int_{\partial\omega} f_x n \, dS = \int_{\omega} r \, dV$$

source term

integrate in time from t_1 to t_2

$$\underbrace{\int_{t_1}^{t_2} dt \frac{d}{dt}\int_{\omega} f_t \, dV}_{=} + \underbrace{\int_{t_1}^{t_2} dt \int_{\partial\omega} f_x \cdot n \, dS}_{} = \int_{t_1}^{t_2} dt \int_{\omega} r \, dV$$

$$\int_{\omega} f_t \, dV \Big|_{t=t_2} - \int_{\omega} f_t \cdot dV \Big|_{t=t_1} + \int_{t_1}^{t_2}\int_{\partial\omega} f_x \cdot n \, (dS\,dt) + \int_{t_1}^{t_2}\int_{\omega} r \, (dV\,dt)$$



$\Omega = \omega \times [t_1, t_2]$

$$\int_{\partial\Omega_t^{+}} f_t \, dS - \int_{\partial\Omega_t^{-}} f_t \, dV + \int_{\partial\Omega_V} f_x \cdot n \, dS = \int_{\Omega} r \, d\bar{V} \quad ⑤$$

Spacetime flux $F = \begin{bmatrix} f_x \\ f_t \end{bmatrix}$ → 1 tensor order higher than f_t

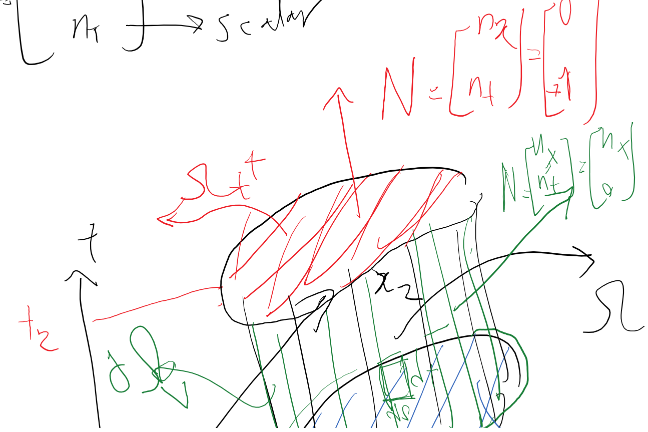
Spacetime normal $N = \begin{bmatrix} n_x \\ n_t \end{bmatrix}$ n_x → vector, n_t → scalar

on $\partial\Omega_t^{+}$ $F \cdot N = \begin{bmatrix} f_x \\ f_t \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = f_t$

$$\int_{\partial\Omega_t^{+}} F \cdot N \, dS = \int_{\partial\Omega_t^{+}} f_t \, dS$$

on $\partial\Omega_t^{-}$

$$\int F \cdot N \, dS \quad \int \begin{bmatrix} f_x \\ f_t \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \cdot \end{bmatrix} dS = -\int f_t \, dS$$



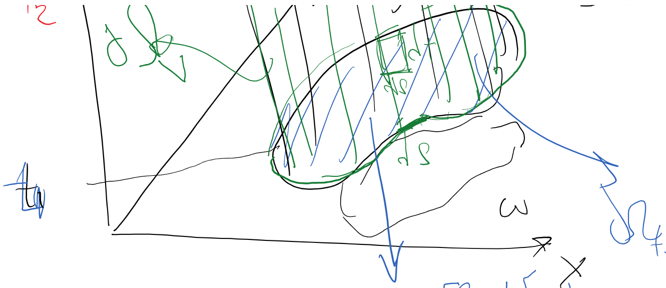
on $\partial\Omega_t^-$

$$\int_{\partial\Omega_t^-} F\cdot N\, dS = \int_{\partial\Omega_t^-} \begin{bmatrix} f_x \\ f_t \end{bmatrix}\cdot\begin{bmatrix} 0 \\ -1 \end{bmatrix} dS = -\int_{\partial\Omega_t^-} f_t\, dS$$

on $\partial\Omega_v$

$$F\cdot N = \begin{bmatrix} f_x \\ f_t \end{bmatrix}\cdot\begin{bmatrix} n_x \\ 0 \end{bmatrix} = f_x n_x$$

$$\int_{\partial\Omega_v} F\cdot N\, dS = \int_{\partial\Omega_v} f_x \cdot n_x\, dS$$



$$N = \begin{bmatrix} n_x \\ n_t \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\partial\Omega = \partial\Omega_t^+ \cup \partial\Omega_t^- \cup \partial\Omega_v$$

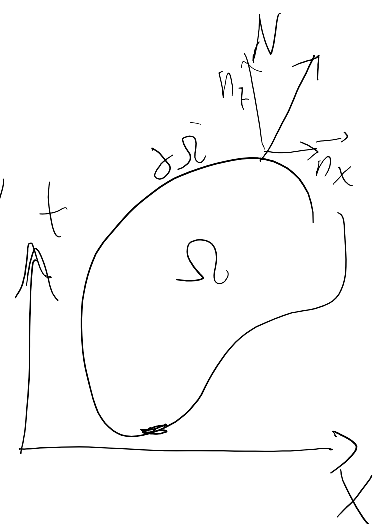
$$\int_{\partial\Omega_t^+} f_t\, dS - \int_{\partial\Omega_t^-} f_t\, dV + \int_{\partial\Omega_v} f_x \cdot n_x\, dS = \int r\, dV$$

$$\int_{\partial\Omega_t^+} F\cdot N\, dS + \int_{\partial\Omega_t^-} F\cdot N\, dS + \int_{\partial\Omega_v} F\cdot N\, dS = \int_{\Omega} r\, dV$$

$$\boxed{\int_{\partial\Omega} F\cdot N\, dS = \int_{\Omega} r\, dV}$$

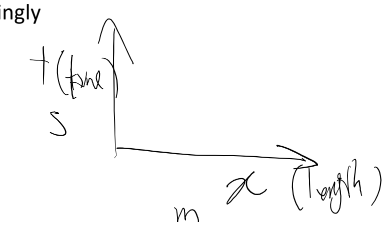
$$F = \begin{bmatrix} f_x \\ f_t \end{bmatrix} \begin{matrix} \to \text{spatial flux} \\ \to \text{temporal flux} \end{matrix} \qquad N = \begin{bmatrix} n_x \\ n_t \end{bmatrix}$$

balance law in spacetime (6)



Although we justify the derivation of (6) from (4)(spatial expression of balance law), (6) is a more general statement of balance law and should be the starting point for deriving (4).

What I did has many problems! Specifically, there is no metric in spacetime; we cannot define normal vectors accordingly



Remedies:

- Multiply time axis by a reference velocity (e.g. light wave speed or any other relevant wave speed to the problem), then all axes have the unit of space and some of these technical difficulties are removed.
- Language of differential forms: This is for example a common approach in relativity,….

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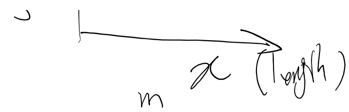
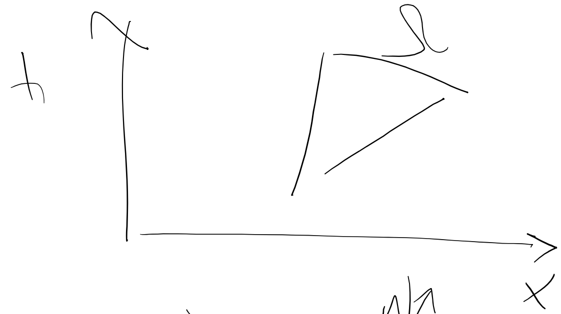


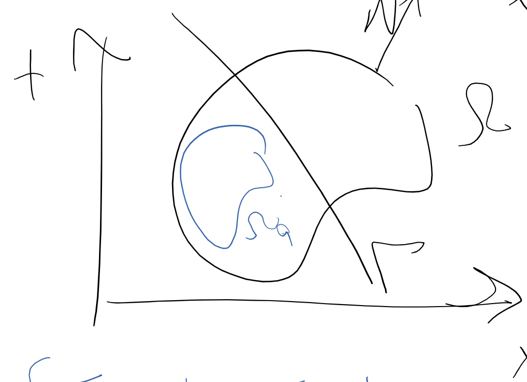
So the expression (6) enables us to consider balance laws for arbitrary shapes in spacetime, which paves the way for formulate FEs in spacetime.



Strong form and jump conditions derived from the balance laws

$$\int_{\partial\Omega} F\cdot N \, ds = \int_{\Omega} r \, dV \quad (\star)$$

balance law



where F is C^1

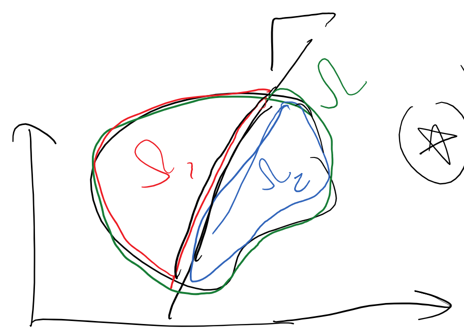
$$\begin{cases} \nabla\cdot F - r = 0 \quad (\text{strong form}) \\ [\![F]\!] = F^-N^- + F^+N^+ = -[\![F]\!]N = 0 \quad \text{on } \Gamma \\ F = F^+ - F^- \quad N = N^- \end{cases}$$

$$\int_{\partial\Omega_a} F\cdot N \, dS = \int_{\Omega_a} r \, dV$$

C^1 function (divergence theorem)

$$\int (\nabla\cdot F - r)\, dV = 0 \rightarrow$$

Ω_a arbitrary $\quad \nabla\cdot F - r = 0$



balance laws

$$(\star)_{\Omega} - (\star)_{\Omega_1} - (\star)_{\Omega_2}$$

$$\int_{\Gamma} (F^-\cdot N^- + F^+\cdot N^+)\, dS = 0 \longrightarrow$$

localization $\longrightarrow$ $[\![F]\!] = F^-N^- + F^+N^+ = 0$ on any manifold

$$-F^-N - F^+N = -\underbrace{(F^+ - F^-)}_{[\![F]\!]}N$$

some other sources call this "exterior minus interior trace" of full flux as jump

We must start from the balance law. Many times the PDEs are written in a non-conservative way. For example, Navier-Stokes equations may be written by dividing certain equations by ρ. In that case we cannot even "guess" what the jump conditions corresponding to the original balance law would look like.