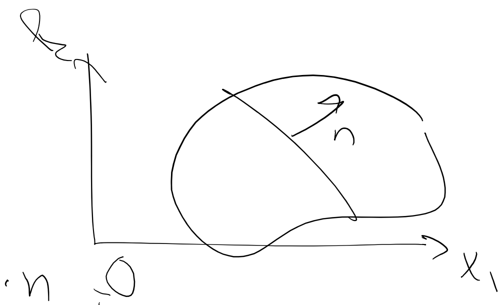
DG2020/03/09

Monday, March 9, 2020 11:41 AM

Some examples for the jump condition



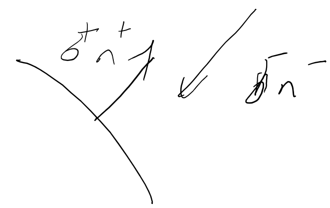
Elastostatics

$f = -\sigma \qquad r = \rho b$

jump condition is $(\sigma^+ - \sigma^-) \cdot n = 0$

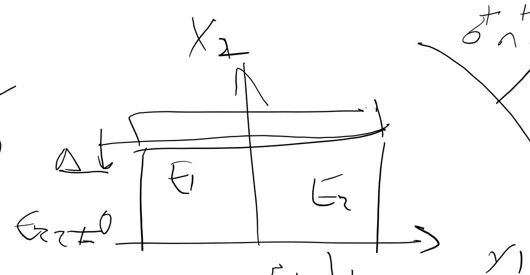
action reaction law $\quad \sigma^+ \cdot n^- = \sigma^- \cdot n^-$

$-\sigma^+ \cdot n^+ = \sigma^- \cdot n^-$



Tractions are equal, but of opposite direction.

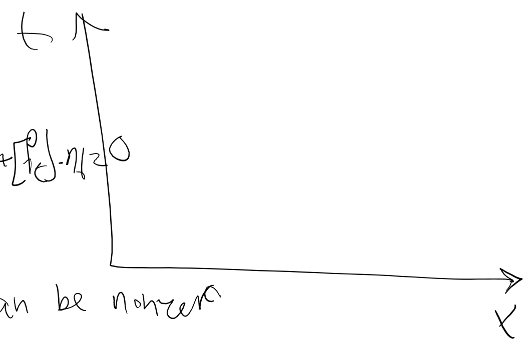
Clearly, we do not have the condition that $\cancel{[\sigma] = 0}$



$[\sigma_{22}] \neq 0$

Jump conditions are more interesting for dynamic problems.

For a dynamic problem:

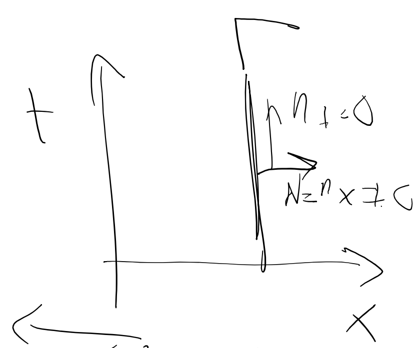


$F = \begin{bmatrix} f_x \\ f_t \end{bmatrix} \qquad [F] \cdot N = 0 \iff [f_x] \cdot n_x + [f_t] \cdot n_t = 0$

So both $[f_x]\, n_x$ & $[f_t]\, n_t$ can be nonzero

Example from Elastodynamics

$f_x = -\sigma$
$f_t = p = \rho v \longrightarrow \boxed{-[\sigma] \cdot n_x + [p]\, n_t = 0}$



$(\rho v)^+ \quad n_t = 1 \quad n_x = 0$

$(\rho v)^-$

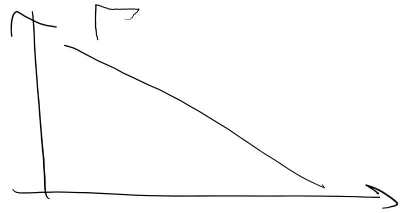
$[p] \cdot n_t = [p] = 0$

$[\rho v] = 0$

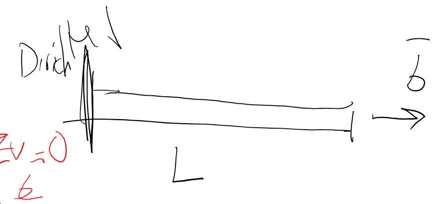
tractions are equal & opposite sign $\qquad [\sigma] \cdot n_x = 0$

1 ——> $[\rho v] = 0$

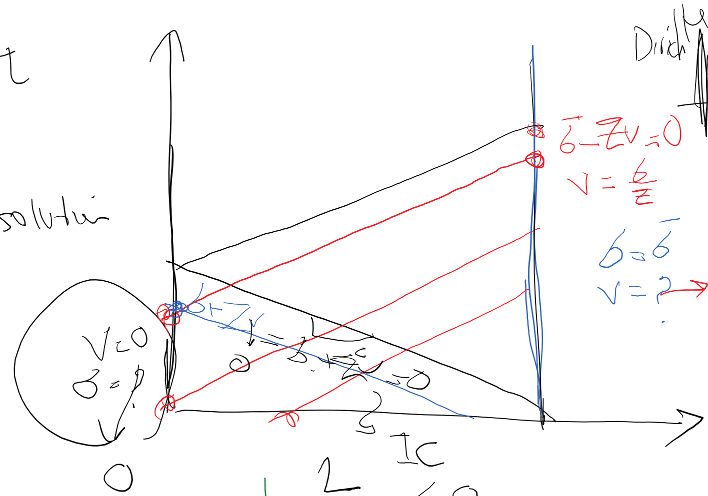
c) $n_x, n_t \neq 0$

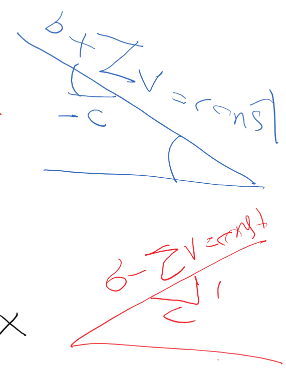


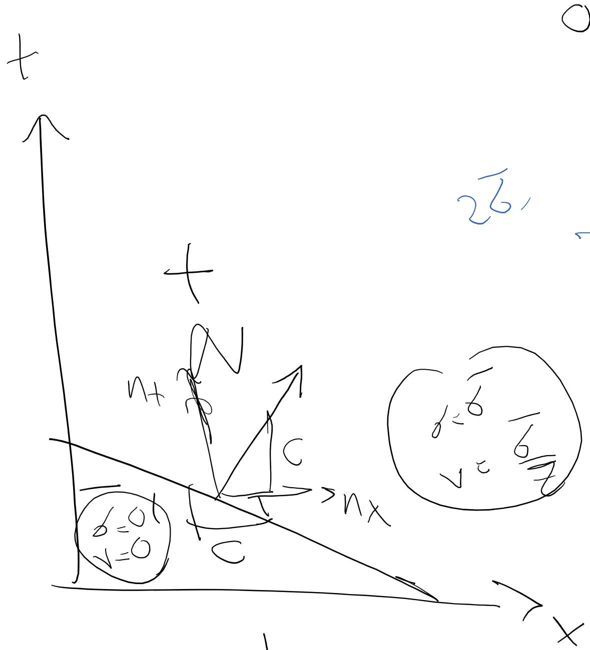
the speed of these jump manifolds is not arbitrary

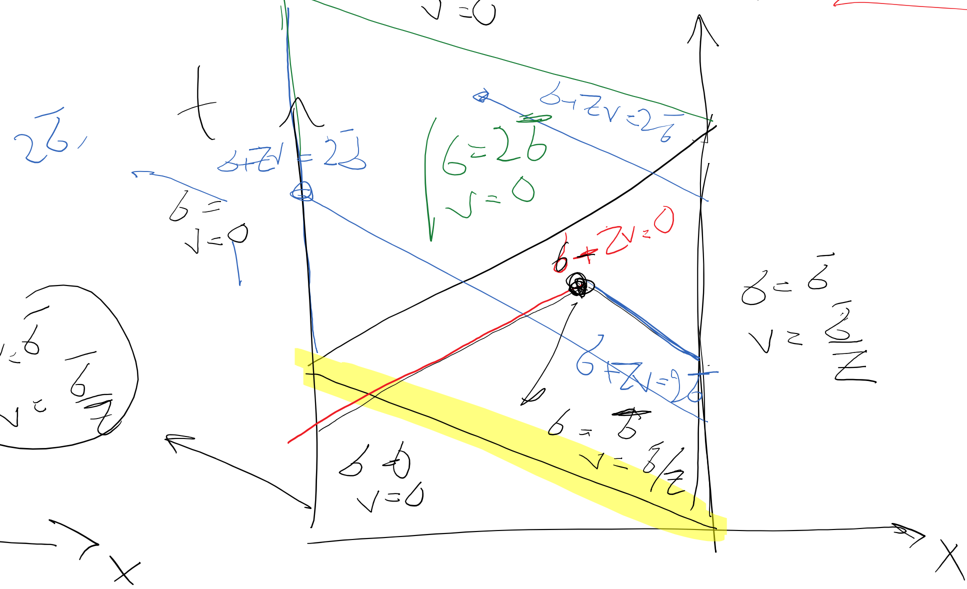


we want to find spacetime solution of σ & v









$$n_x = \frac{1}{\sqrt{1+c^2}}$$

$$n_t = \frac{c}{\sqrt{1+c^2}}$$

$$[\sigma] = \bar\sigma$$

$$[v] = \frac{\bar\sigma}{Z}$$

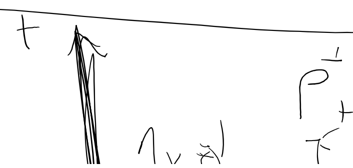
$$[\vec{p}] = \rho \frac{\bar\sigma}{Z}$$

$$-[\sigma] \cdot n_x + [p] n_t =$$

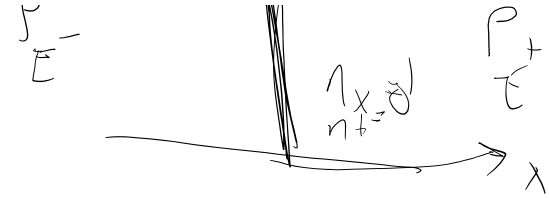
$$-\bar\sigma \frac{1}{\sqrt{1+c^2}} + \frac{\bar\sigma \rho c}{Z} \frac{}{\sqrt{1+c^2}} = 0$$

$$= 0 \checkmark$$

Material interfaces



Material interface



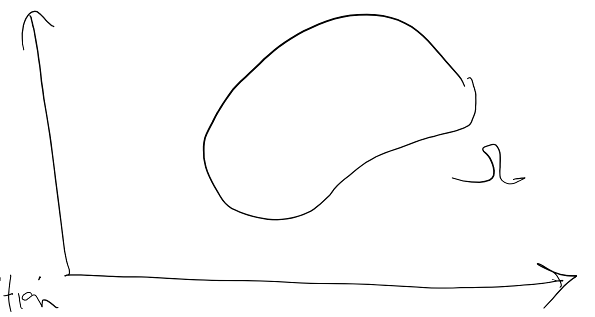
only tractions are continuous $\quad -[\sigma]\cdot n_x = 0$

$[p]$ can be nonzero ($[v]=0$ by continuity) but if $\rho^- \neq \rho^+$ then $[p] \neq 0$)

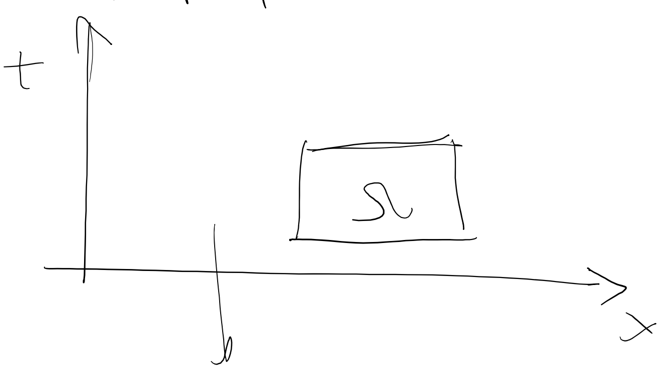
Formulating a spacetime discontinuous Galerkin method:

$\nabla_{st} \cdot F - r = 0 \quad$ strong form PDE

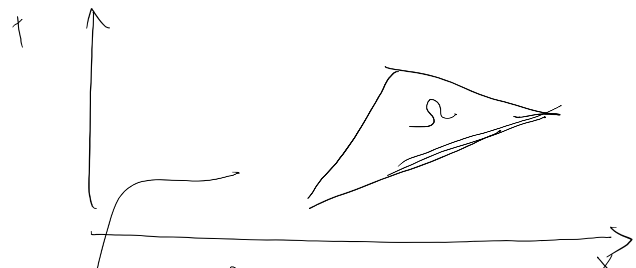
$[F]\cdot N = [f_x]\cdot n_x + [f_t]\, n_t = 0 \quad$ jump condition



$\longrightarrow$ for spacetime methods we have two general classes of elements

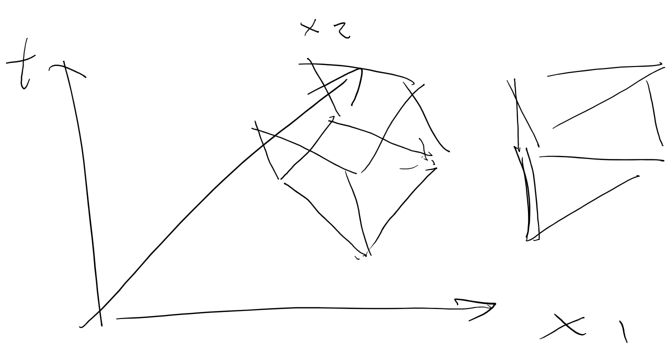


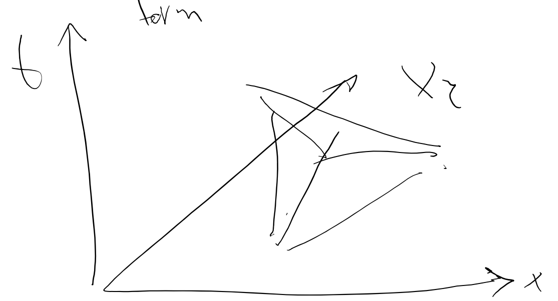
similar to an implicit scheme



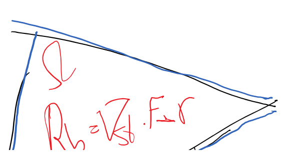
simplicial elements in spacetime

something similar to explicit schemes will form



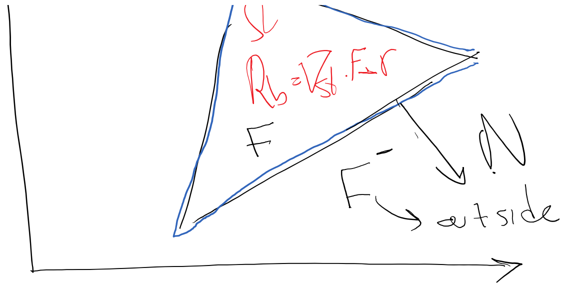


$R_b = \nabla_{st} \cdot F - r$



$R_b = V_{st} \cdot F - r$

$R_g = [F] \cdot N$



$[F] = [F^- - F] \cdot N$

$[F^{out} - F^{in}] \cdot N = 0$

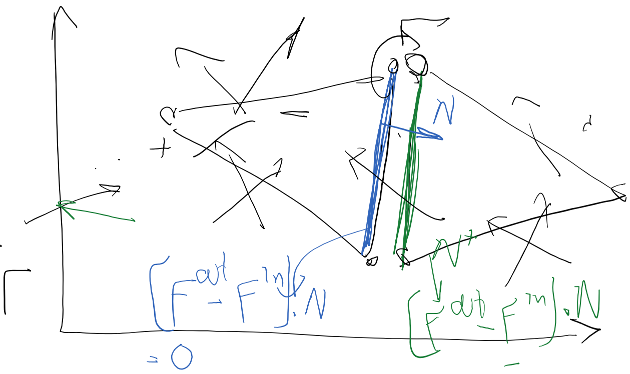
exterior trace of element

interior trace

$\boxed{[\cancel{F}^{out} - F] \cdot N = 0}$

$[F^*(F^{in}, F^{out}) - F] \cdot N = 0$

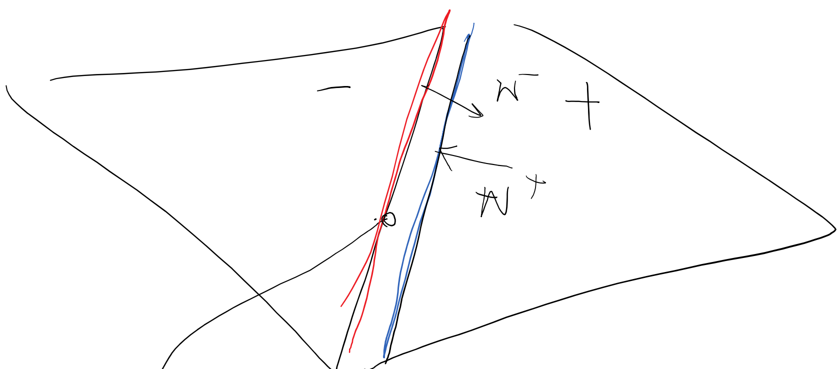
e^- e^+ are coupled through Γ



$[F^{out} - F^{in}] \cdot N = 0$

$[F^+ - F^-] \cdot N^- = 0$

$[F^{out} - F^{in}] \cdot N = [F^- - F^+] \cdot N^+ = 0$



$[F^{*+}(F^-, F^+) - F^-] \cdot N^- = 0$ (1)

$[F^*(F^-, F^+) - F^+] \cdot N = 0$ (2)

if $F^{*-} = F^{*+} = F^*$

almost always the case

add them $\Downarrow$

$[F^+ - F^-] \cdot N^- = 0$

F^* is a physically correct solution that is determined from the conditions of the interface on left & right.

— ($*$) option can be used to define many interesting target values

fracture

always

FD: $F_{\text{individual}}\begin{bmatrix}\sigma \\ V\end{bmatrix}$

$[\sigma].n_x = 0$

$[V] \neq 0$ for example if we have crack opening or frictional slide

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the equations are

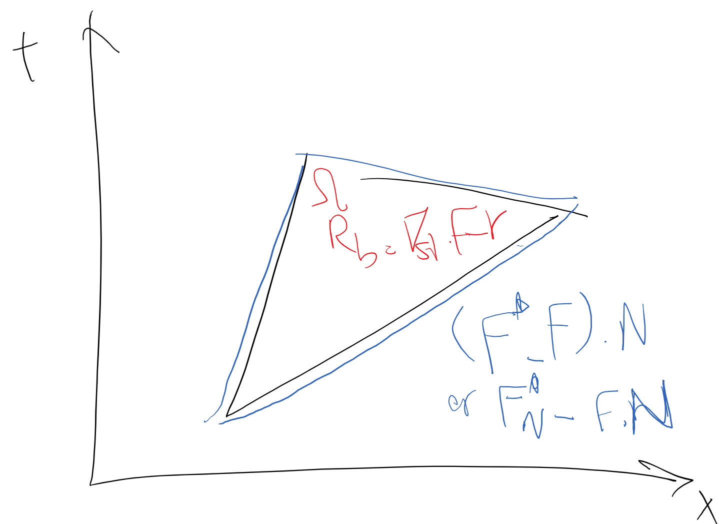
$$R_b = \nabla_{st} \cdot F - r$$

$$R_\gamma = (F^* - F).N$$

or $\underline{F^*_N} - F.N$

F is interior trace

$F^*(F, F^{\text{outside}})$



So, the use of star values in the context of FV and DG methods is more flexible than simply writing (F+ - F-).N = 0. It provides many options for the definition of numerical flux (for example average, Riemann and various forms of approximate Riemann fluxes) and physical conditions at an interface (contact, friction, resistive sheet, …)

Continuum wR& weak statement.

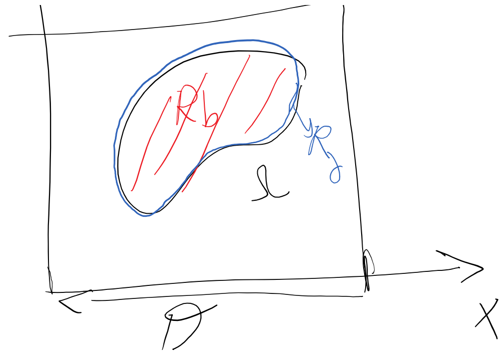
F is the weak solution if for any $\Omega \subseteq \Omega \times [0, T]$

$\forall w$ & $\forall \Omega$

$\forall \omega$ & $\forall \Omega$

$$\int_\Omega \underbrace{\left(\omega\left(\nabla_{st}\cdot F - r\right)\right)}_{R_b} dV + \int_{\partial\Omega} \omega\underbrace{\left(\hat{F}_n - F\cdot n\right)}_{R_\partial} dS = 0 \qquad (1)$$

ω: weight



We use the Gauss theorem to get to the weak statement.

$$\omega\nabla_{st}\cdot F = \nabla_{st}\cdot \omega F - \nabla_{st}\omega\cdot F$$

$$\int_\Omega \omega\nabla_{st}\cdot F\, dV = \int_\Omega \nabla_{st}\cdot\omega F\, dV - \int_\Omega \nabla_{st}\omega F = \int_{\partial\Omega} \omega F\cdot N\, dS - \int_\Omega \nabla_{st}\omega F$$

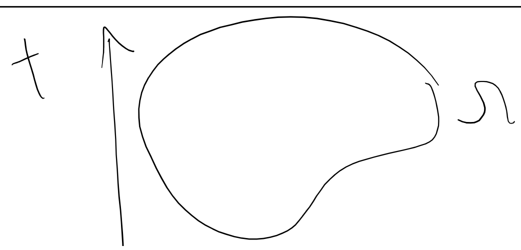
plug this into (1)

$$\underbrace{\left(\cancel{\int_{\partial\Omega}\omega F\cdot N\, dS} - \int_\Omega \nabla_{st}\omega F\right)}_{\int_\Omega \omega\nabla_{st}\cdot F} - \int_\Omega \omega r\, dV + \int_{\partial\Omega}\omega\hat{F}_n\, dS - \cancel{\int_{\partial\Omega}\omega F\cdot N\, dS} = 0$$

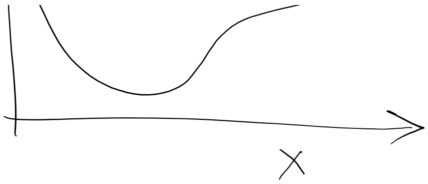
continuous weak statement is

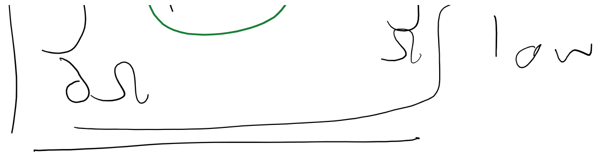
$\hat{F}\cdot \tilde{N}$

$$\boxed{\forall\Omega,\ \forall\omega\in V \qquad \int_\Omega\left(-\nabla_{st}\omega\cdot F - \omega r\right)dV + \int_{\partial\Omega}\omega\hat{F}_n\, dS = 0}$$



(a) $\displaystyle \int_{\partial\Omega} F\cdot N\, dS = \int_\Omega r\, dV$ balance law





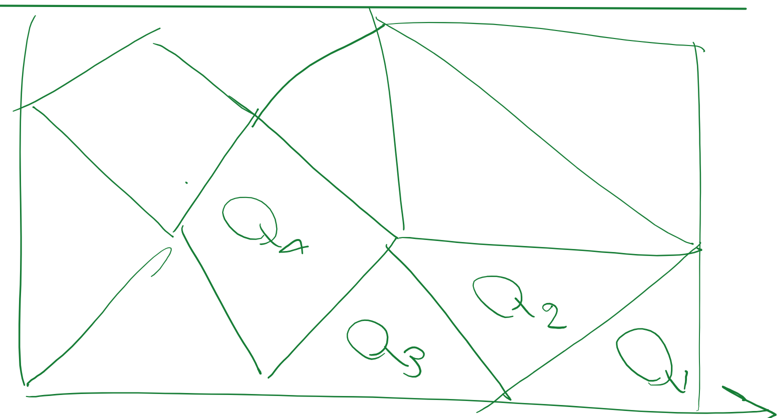
$\omega = 1 \quad \nabla_{st}\,\omega = 0 \longrightarrow -\int r\, dV + \int F_n^{*}\, dS = 0$

$\Rightarrow \boxed{\int_{\partial\Omega} F_n^{*}\, dS = \int_{\Omega} r\, dV}$

We satisfy the balance law w.r.t. target values

Now we want to go to discrete setting

$\forall\, \Omega \rightarrow$ spacetime mesh



$\Omega \rightarrow$ spacetime elements