DG2020/03/11

Wednesday, March 11, 2020 11:41 AM

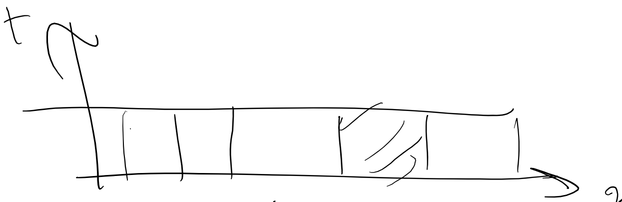
Interpolation of shape functions:
Remember that for time marching schemes, the solution was interpolated as,

$$T(x,t) = \sum_{i=1}^{n} T_i(x)\, a_i(t) \longrightarrow M\ddot{a} + C\dot{a} + Ka = F$$

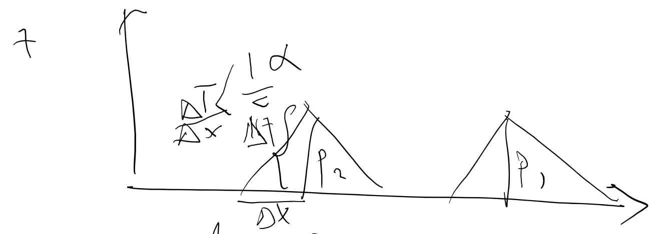
For spacetime methods

$$\boxed{T(x,t) = \sum_{i=1}^{n} T_i(x,t)\, \underbrace{a_i}_{\text{unknown dofs}} \longrightarrow Ka = F}$$

No mass or damping matrices

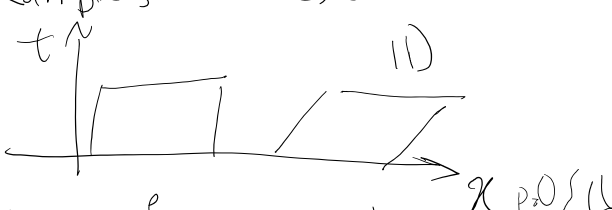


similar to implicit time marching
"the slab needs to be solved as a coupled system"



Causal SDG
similar to explicit time marching methods

Examples of basis functions



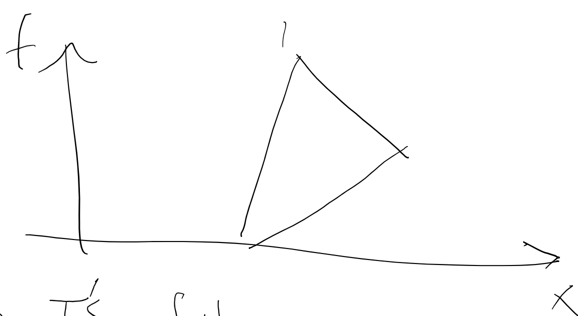
$p=0$ $\{1\}$

product of x & t basis

$p=1$ $\begin{bmatrix} 1 \\ x \end{bmatrix} \otimes \begin{bmatrix} 1 \\ t \end{bmatrix}$ $\boxed{\begin{matrix} 1 & t \\ x & xt \end{matrix}}$

$P=2$ $\begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ t \\ t^2 \end{bmatrix}$ $\boxed{\begin{matrix} 1 & t & t^2 \\ x & tx & t^2x \\ x^2 & tx^2 & t^2x^2 \end{matrix}}$

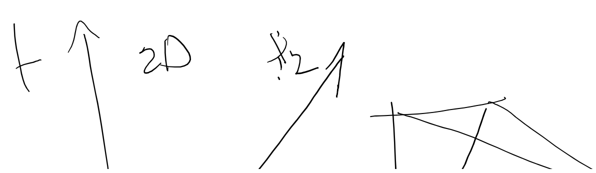
1D

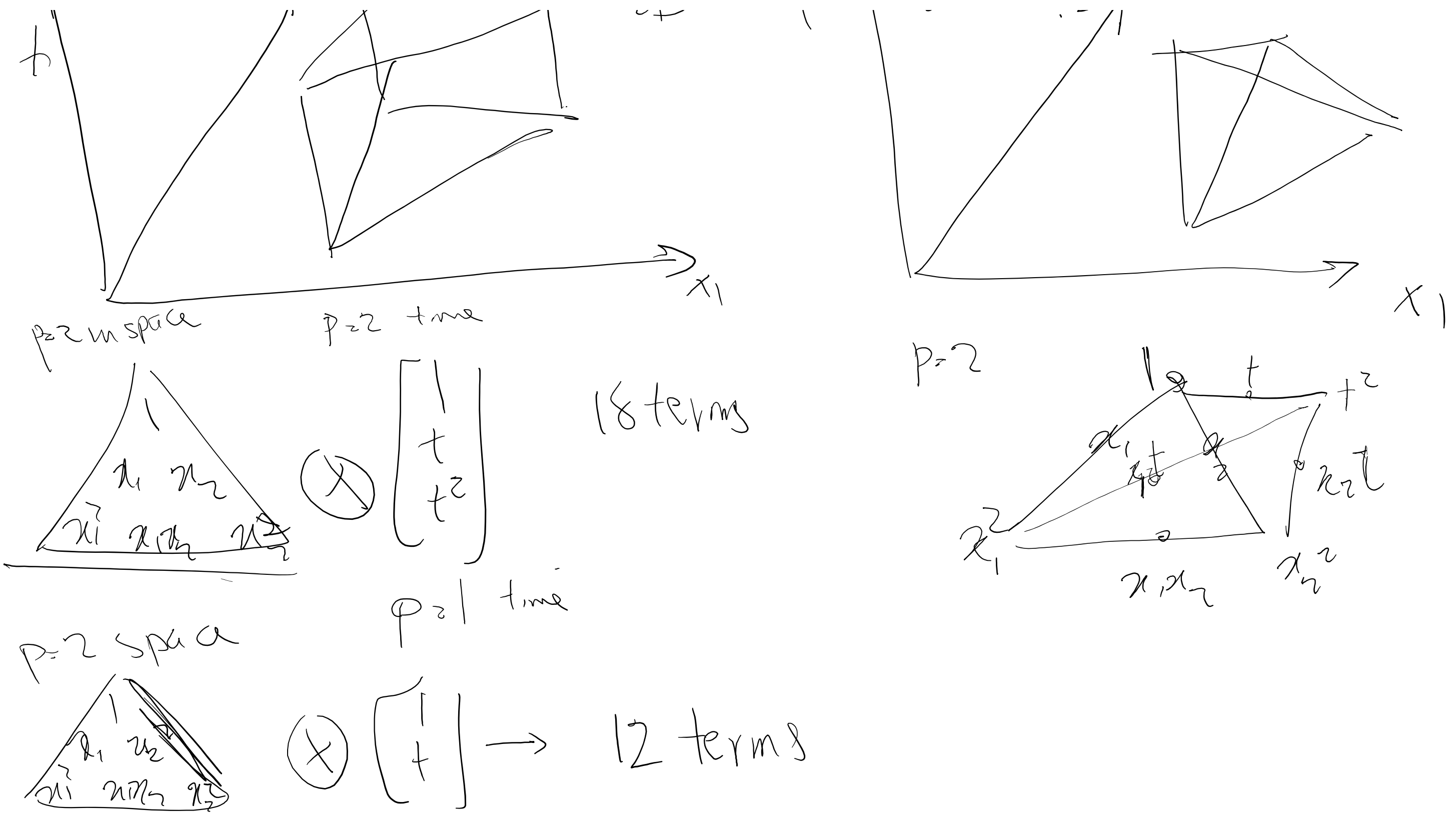


$P=0$ T's $\{1\}$

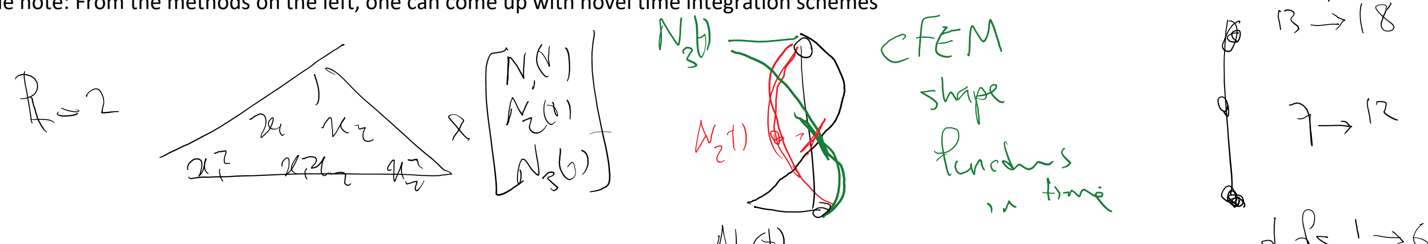
$P=1$
$$\begin{matrix} & 1 & \\ x & & t \end{matrix}$$

$P=2$
$$\begin{matrix} & & 1 & & \\ & x & & t & \\ x^2 & & xt & & t^2 \end{matrix}$$





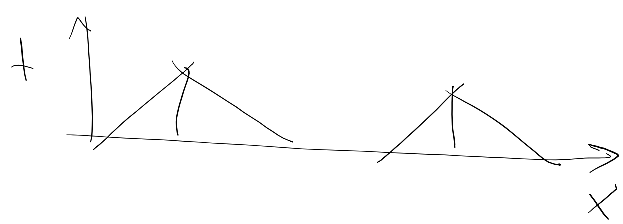
Side note: From the methods on the left, one can come up with novel time integration schemes



If we follow up a formulation with this type of spatial mesh extrusion in time and basis function in time, we can come up for high order time integration schemes for time marching methods.

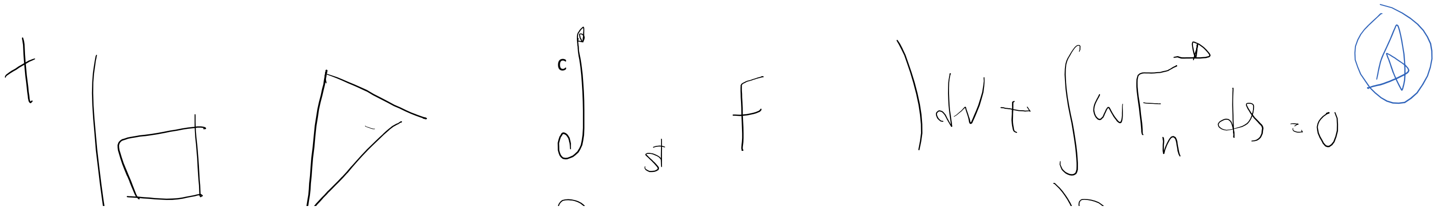
In fact, by using shape functions like this and integration of the balance law in spacetime and getting rid of time dependencies, we get something similar to RK implicit methods, etc. There are many examples in the literature that time integration schemes are derived this way for a particular problem.

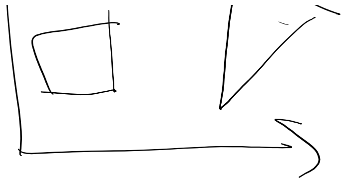
We'll mostly focus on "causal" spacetime meshes



We'll do two sample formulations for the thermal heat conduction and elastodynamics.

General weak statement:



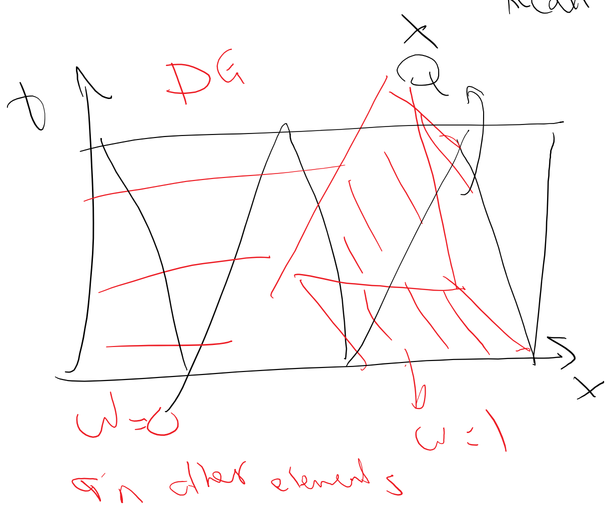


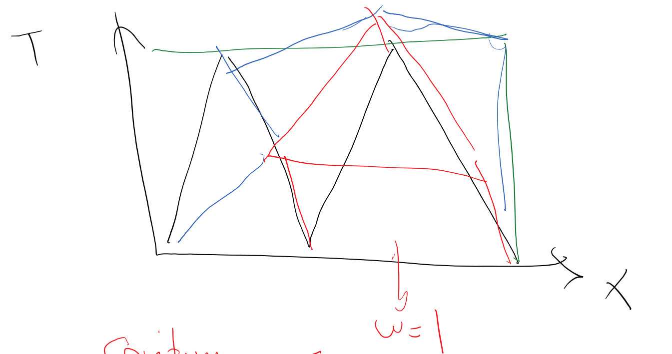
$$\ldots \, dV + \int_{\partial Q} w F_n \, ds = 0$$

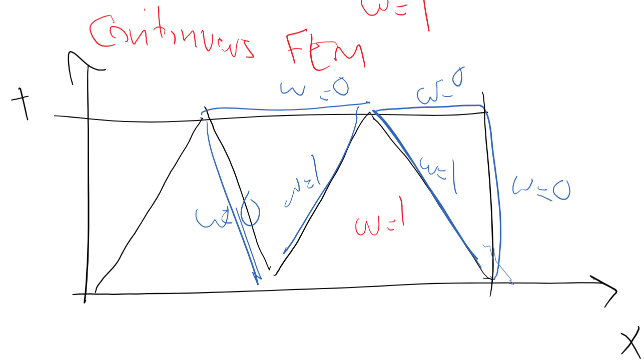
Recall $w = 1$

$$\boxed{\int_{\partial Q} F_n^{*} \, ds = \int_Q r \, ds}$$

For DG methods we can recover balance laws @ the element level







For continuous FEMs we cannot isolate and element and have w = 1 only in that element. Because of this, for cFEMs we can only ensure the satisfaction of balance laws for the entire (spatial / spacetime) domain by setting w = 1 everywhere.

For DG methods, we can set w = 1 only in one element and recover the balance laws per element. This is a selling point for DG methods. Satisfaction of balance laws at the element level is highly desired and very important in many applications (electromagnetics balance of charges, …)

Expansion of ⊛

$$\int_Q \left(-\nabla_{st} w \cdot F - w r \right) dV + \int_{\partial Q} w F_n^{*} \, ds = 0 \qquad \nabla_{st} w = \begin{bmatrix} \nabla w \\ \dot{w} \end{bmatrix}$$

$$\int_Q \left(-\begin{bmatrix} \nabla w \\ \dot{w} \end{bmatrix} \cdot \begin{bmatrix} f_x \\ f_t \end{bmatrix} - w r \right) dV + \int_{\partial Q} w \begin{bmatrix} f_{x_n}^{*} \\ f_{t_n}^{*} \end{bmatrix} ds = 0 \qquad F = \begin{bmatrix} f_x \\ f_t \end{bmatrix} \qquad F_n^{*} = \begin{bmatrix} f_{x_n}^{*} \\ f_{t_n}^{*} \end{bmatrix}$$

$$\boxed{\int_Q \left(-\nabla w f_x - \dot{w} f_t - w r \right) dV + \int_{\partial Q} w \left(f_{x_n}^{*} + f_{T_n}^{*} \right) ds = 0}$$

$$F^{*} \cdot N \longrightarrow w \left(f_x^{*} n_x + f_t^{*} \cdot n_t \right)$$

$$f_n^{*} = F^{*} \cdot N \longrightarrow \omega\left(f_x^{*} n_x + f_t^{*} n_t\right)$$

a lot of times f_n^{*} is obtained from F^{*} multiplied by N

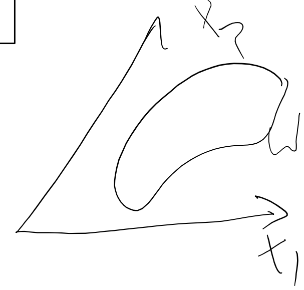
(spatial DG $q^{*} = \{q\}$ $\quad q_n^{*} = q^{*} \cdot n$

Expanded weak statement

Example 1:
Thermal problem:

$$F = \begin{bmatrix} q \\ CT \end{bmatrix} \begin{matrix} \rightarrow \text{spatial} \\ \rightarrow \text{temporal} \end{matrix}$$

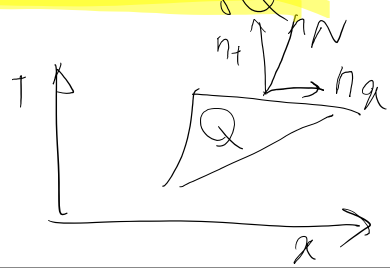
$$\underbrace{\frac{d}{dt}\int_{\omega} CT \, dV}_{f_T} = -\underbrace{\int_{\partial\omega} q \cdot n \, ds}_{f_x} + \underbrace{\int_{\omega} Q \, dV}_{r}$$



$$F^{*} N = F_N^{*}$$

$$\text{WR} \quad \int \omega\left(\nabla_{\bar{x}} F - r\right) dv + \int_{\partial Q} \omega (F^{*} - F) \cdot N \, ds = 0 \rightarrow \int_{Q} \omega\left(C\dot{T} + \nabla \cdot q - Q\right) dv + \int_{\partial Q} \left[\omega (q^{*} - q) \cdot n_x + \omega (CT^{*} - CT) n_t\right] ds = 0$$

$$\text{WK:} \quad \int \left(-\nabla_{\bar{x}} F - \omega r\right) dv + \int_{\partial Q} \omega F^{*} \cdot N \, ds = 0 \longrightarrow \int_{Q} \left(-\dot{\omega} CT - \nabla\omega \cdot q - \omega Q\right) dv + \int_{\partial Q} \left(\omega q^{*} \cdot n_x + CT^{*} n_t\right) ds = 0$$

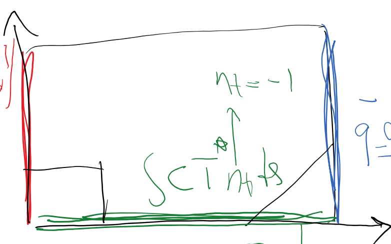


(2)

problem with (2)

$$\int_{\partial Q} \left(\omega q^{*} n_x + CT^{*} n_t\right) ds$$

for vertical boundary on ∂D_T (BC is specified)



$$\int_{\partial Q} \left(\omega q^{*} n_x\right) ds \quad (n_t = 0)$$

IC need to specify

$CT^{*} = CT(x, t=0)$ ✓

can be done

Two ways to fix the issue with vertical interfaces (and in general a term multiplying nx)

① Modify $q^* = \{q\} \longrightarrow q^* = \{q\} + \hat{\alpha}[T]$ $\hat{\alpha} = \frac{\alpha}{h}$

we talked about this before.

Recall from prior discussion that in many DG formulations for elliptic PDEs we needed this penalty term for stability. For parabolic PDEs, we don't have to have this. You can refer to all those papers that discuss different forms of numerical fluxes for parabolic PDEs

> New Volume (D:) > MyDocuments > acd > courses > DG > DG_course > Papers > Fluxes > ParabolicFluxes

Name

- Baumann_1999_A discontinuous hp FEM.pdf
- Shu_2001_49_Different formulations of the discontinuous Galerkin method for the viscous terms.pdf
- Zhang_2003_Shu_44_An analysis of three different formulations of the discontinuous Galerkin method for diffusion equations.pdf

We don't have the penalty term added for q*.

② Basically add a jump term to the boundary

now

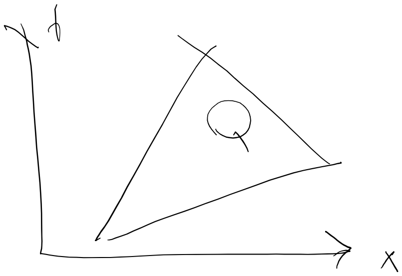
$$\text{WRS}: \int_\Omega \hat{T}\,(\rho \dot{T} + \nabla\cdot q - r)\,dv + \int_{\partial\Omega} \hat{T}\left[(q^* - q)\cdot n_x + (T^* - T)\,n_t\right] ds$$

$$\int_\Omega \hat{q}\,(k^{-1}q + \underbrace{\nabla T}_{\text{spatial derivative}})\,dv + \int_{\partial\Omega} \hat{q}\cdot n\,(T^* - T)\,ds = 0$$

2 field formulation for T, q

$T =$ (1; x t; x^2 xt t^2) $\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ \vdots \\ T_6 \end{bmatrix}$ $q = \begin{bmatrix} q \\ q_x \end{bmatrix} =$ (1; x t; x^2 xt t^2) $\begin{bmatrix} q_1 \\ \vdots \\ q_6 \end{bmatrix}$

$p = 2$



element has 12 dofs $k^{-1}q - \nabla T \neq 0$ $\int_\Omega \hat{q}\,(\underbrace{k^{-1}q + \nabla T}) dv$ weakly satisfies it.

$\overline{2F} \longrightarrow 1F$ $q = -k\nabla T \longrightarrow$ get rid of this

$$\int_\Omega \hat{q}\,(k^{-1}q + \nabla T)\,dv = 0$$

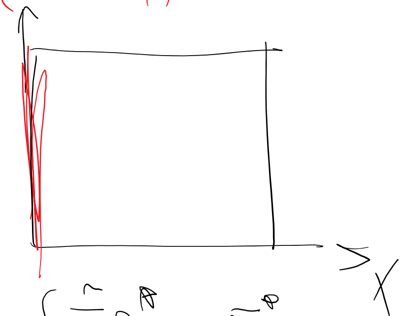
WRS ...

WRS

$$\int_Q \hat{T}(C\dot{T} + \nabla\cdot q - r)\,dv + \int_{\partial Q} \hat{T}\left[(q^* - q)\,n_x + (T^* - T)\,n_t\right] ds$$

$$+\,\epsilon \int_{\partial Q} \hat{q}\cdot n_x\,(T^* - T)\,ds = 0$$

can be specified



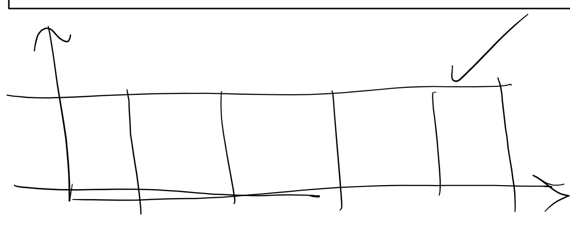
$T^* = \bar{T}$

(3)

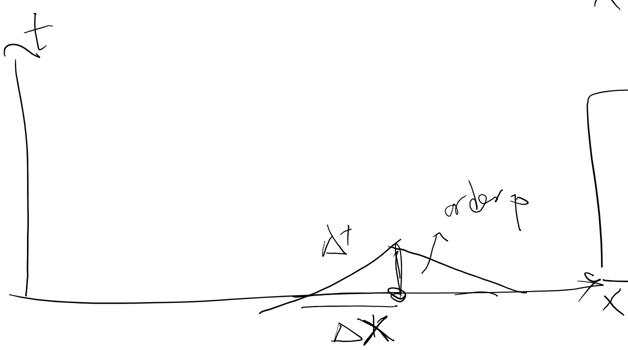
WK

$$\int_Q \left(-\dot{\hat{T}}\,CT - \nabla\hat{T}\,q - \hat{T}r\right) dv + \int_{\partial Q} \hat{T}\,q^* n_x + \hat{T}\,T^* n_t$$

$$+\,\epsilon \int \hat{q}\cdot n_x\,(T^* - T)\,ds = 0$$



extruded meshes in time can solve the above parabolic PDE



$$\boxed{\Delta t = C(P)\,\frac{\Delta x^2}{D}}$$

$$C\dot{T} - \nabla\cdot K\nabla T = 0 \qquad K \text{ constant scalar}$$

$$\dot{T} - \frac{K}{C}\,\Delta T = 0, \qquad D = \frac{K}{C}$$

$$\dot{T} - D\,\Delta T = 0$$

$$[D] = \frac{L^2}{T}$$

How about hyperbolic heat conduction?

$$C\dot{T} - \nabla\cdot K\nabla T = Q \qquad C\dot{T} - K\Delta T = Q + \nabla K\cdot\nabla T \quad \text{if}$$

1D $\quad C\dot{T} - K\,T_{,xx}$ - - . —

lower order

$$A\,u_{,xx} + B\,u_{,xy} + C\,u_{,yy} = f(u_{,x}, u_{,y}, u) \quad \text{if } k=0$$

$$= Q$$

$$\begin{bmatrix} \partial/\partial x & \partial/\partial y \end{bmatrix} \underbrace{\begin{bmatrix} A & B/2 \\ B/2 & C \end{bmatrix}}_{\mathcal{A}} \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix} u = \;''$$

$$\det \mathcal{A} = AC - \frac{B^2}{4} \begin{cases} <0 & \text{hyperbolic} \\ =0 & \text{parabolic} \\ >0 & \text{elliptic} \end{cases}$$

$$C\dot{T} - k\,u_{,xx} = \_\_$$

$$C = -k \longrightarrow \Delta = 0 \quad \text{parabolic}$$

$$A = B = 0$$

Maxwell-Cattaneo-Vernotte (MCV) heat model

$$\underbrace{\tau C}_{\bar{A}}\ddot{T} + C\dot{T} - \underbrace{k\,T_{,xx}}_{\bar{C}} = Q + \tau\dot{Q}$$

$$\bar{B} = 0$$

$$\bar{A}\bar{C} - \left(\frac{\bar{B}}{2}\right)^2 = -\tau C k \;\cancel{<}\; 0 \quad \text{hyperbolic}$$

wave speed

damping

$$A\ddot{T} + D\dot{T} = C\,T_{,xx} \ldots$$

$$c = \sqrt{\frac{C}{A}} \quad \text{wave speed}$$

$$Z = cA = \sqrt{CA} \quad \text{impedance}$$

How to form a hyperbolic heat conduction model?
There are numerous hyperbolic heat conduction models in the literature. We'll just cover one here

spatial gradients

MCV

Fourier heat model (Fickian model)

$$C\dot{T} + \nabla\cdot q = Q$$

Fourier heat law ← $q = -K\nabla T$

Fickian diffusion model

$$C\dot{T} + \nabla\cdot q = Q$$

$$\tau\dot{q} + \nabla\cdot KT = -q + (\nabla\cdot K)\cdot T$$

time rates

conductivity matrix

lower order derivatives

systems of conservation laws

MCT

single lag (relaxation model)

$$\tau\dot{q} + q = -K\nabla T \qquad \boxed{\tau\dot{q} = -(q + K\nabla T)}$$

$\tau = 0$ $\quad \nabla T$ given $\quad q = -K\nabla T$

$\tau > 0$ there is a lag $\quad \nabla T$ fixed is given

q starts from its current value & tends to $-K\nabla T$

in time scales comparable to relaxation time τ