DG2020/03/23

Monday, March 23, 2020 11:40 AM

From last time: MCV equation for heat conduction:

$$\left.\begin{aligned} C\dot{T} + \nabla\cdot q &= Q \\ q &= -K\nabla T \end{aligned}\right\} \longrightarrow \left\{\begin{aligned} &C\dot{T} + \nabla\cdot q = Q \\ &\underbrace{\tau\dot{q}}_{\text{Relaxation term}} + \underbrace{\nabla\cdot KT}_{\text{full divergence}} = -q + \underline{\nabla\cdot K\,T} \quad (1) \end{aligned}\right.$$

Fourier heat law

$$\nabla\cdot KT = (K_{ij}T)_{,j} = K_{ij,j}T + K_{ij}T_{,j} = \nabla\cdot K\,T + K\nabla T$$

$$\rightarrow \boxed{K\nabla T = \nabla\cdot KT - (\nabla\cdot K)T}$$ used in eqn (1)

if K = const (simplifying assumption)

$$\left\{\begin{aligned} &C\dot{T} + \nabla\cdot q = Q \\ &\tau\dot{q} + \nabla\cdot KT = -q \end{aligned}\right. \longrightarrow \quad (\dot{\ })^2\left(C\dot{T} + \nabla\cdot\dot{q} = \dot{Q}\right)\times\tau$$

$$\nabla\cdot \qquad \tau\nabla\cdot\dot{q} + \nabla\cdot\nabla\cdot KT = -\nabla\cdot q$$

$$\boxed{C\tau\ddot{T} + C\dot{T} - \nabla\cdot K\nabla T = Q + \tau\dot{Q}}$$

hyperbolic PDE

$$\text{wave speed} = \sqrt{\frac{\max\text{ eigenvalue}(K)}{C\tau}}$$

if K is diagonal (isotropic conduction)

$$\boxed{\underset{\text{wave speed}}{c} = \sqrt{\frac{K}{C\tau}}}$$

$$\bar{\bar{K}} = K\,I$$

MCV heat eqn written as a system of conservation law

$$\begin{aligned} C\dot{T} &+ \nabla\cdot q = Q \\ \tau\dot{q} &+ \nabla\cdot KI = -q + (\nabla\cdot K)T \end{aligned}$$

$$\Sigma\dot{q} + \nabla\cdot KT = -q + (\nabla\cdot K)T$$

time $\quad \nabla\cdot \quad$ source terms

state

$$f_t + \nabla\cdot f_x = r$$

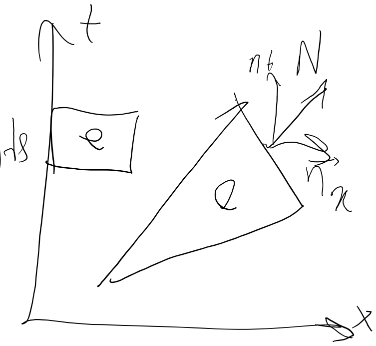
$$f_t = \begin{bmatrix} CT \\ \Sigma q \end{bmatrix}_{\text{vector}}^{\text{scalar}} \qquad f_x = \begin{bmatrix} q \\ KT \end{bmatrix} \begin{matrix} \to \text{vector} \\ \to \text{2nd order tensor} \end{matrix} \qquad r = \begin{bmatrix} Q \\ -q + (\nabla\cdot K)T \end{bmatrix}$$

$$(KT)_{ij} = K_{ij}T$$

WRS

$$0 = \int_e \omega\left(\dot{f}_t + \nabla\cdot f_x - r\right) dv + \int_{\partial e} \omega\left[(f_t^* - f_t)n_t + (f_x^* - f_x)n_x\right] ds$$

$$\int_e \omega\left(\nabla_{s+1}\cdot F - r\right) dv + \int \omega\,(F^* - F)\cdot N\, ds = 0 \qquad F = \begin{bmatrix} f_x \\ f_t \end{bmatrix}$$



(3)

$$f_t = \begin{bmatrix} CT \\ \Sigma q \end{bmatrix} \longrightarrow \omega = \begin{bmatrix} \hat{T} \\ \hat{q} \end{bmatrix} \qquad f_x = \begin{bmatrix} q \\ KT \end{bmatrix}$$

WRS:

$$\int \begin{bmatrix} \hat{T} \\ \hat{q} \end{bmatrix} \begin{pmatrix} C\dot{T} + \nabla\cdot q - Q \\ \Sigma\dot{q} + K\nabla T + q \end{pmatrix} dv + \int_{\partial e} \begin{bmatrix} \hat{T} \\ \hat{q} \end{bmatrix} \begin{pmatrix} ((CT)^* - (CT))n_t + (q^* - q)n_x \\ ((\Sigma q)^* - (\Sigma q))n_t + (KT^* - KT)n_x \end{pmatrix} ds = 0$$

WK

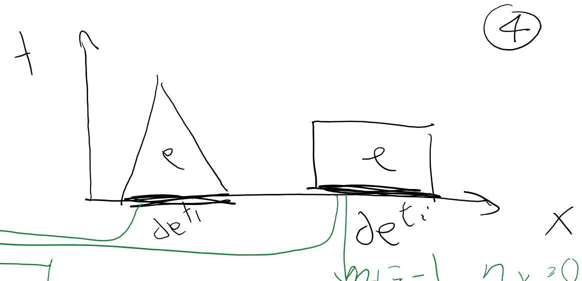
$$\int \begin{bmatrix} -\dot{\hat{T}}CT - \nabla\hat{T}q - \tilde{T}Q \\ -\dot{\hat{q}}\,\Sigma q - \nabla\hat{q}\,KT + \hat{q}q \end{bmatrix} + \int_{\partial e} \begin{bmatrix} \hat{T}\left(CT^* n_t + q^* n_x\right) \\ \hat{q}\left((\Sigma q^*)n_t + KT^* n_x\right) \end{bmatrix} ds = 0$$

(4)

IC needed here

we need T^* & q^*

for IC



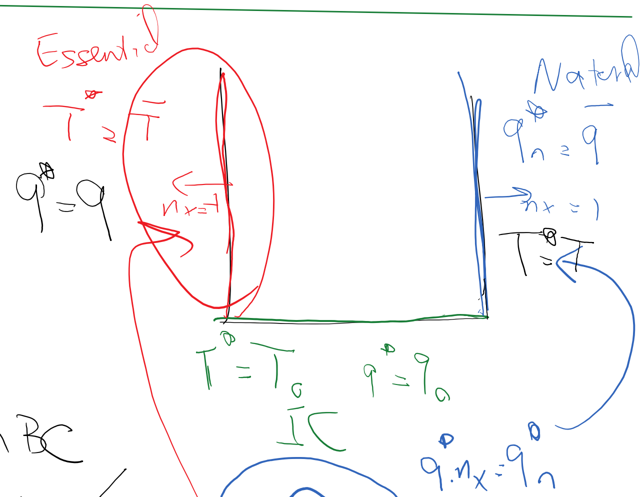
$$T^*(x) = T_0(x)$$

$$\boxed{\begin{aligned} T^*(x) &= T_0(x) \\ q^*(x) &= q_0(x) \end{aligned} \quad \text{at } t=0 \quad \text{IC}}$$

$n_t = -1 \quad n_x = 0$

Note for parabolic heat conduction we only had $T^*(x) = T_0(x)$

How about boundary conditions?



on BC

$$\int \begin{bmatrix}\hat{T}\\ \hat{q}\end{bmatrix}\begin{pmatrix} C\dot{T} + \nabla\cdot q - Q \\ \tau\dot{q} + k\nabla T + q\end{pmatrix} dv + \int_{e}\begin{bmatrix}\hat{T}\\ \hat{q}\end{bmatrix}\begin{bmatrix}\left((CT)^* - (CT)\right)n_t + (q^* - q)n_x \\ \left((\tau q)^* - (\tau q)\right)n_t + (kT^* - kT)n_x\end{bmatrix} ds = 0$$

Recall from the last time that for parabolic heat equation, we could not specify T* on BC:

$$\int_e w(C\dot{T} + \nabla\cdot q - Q)\,dv + \int w(q_n^* - q\cdot n_x) + (CT^* - CT)n_t\, ds = 0$$

no $T^* n_x$ term

Is there anyway to recover a decent parabolic formulation from the hyperbolic MCV WRS?

$\tau = 0$

$$\int \begin{bmatrix}\hat{T}\\ \hat{q}\end{bmatrix}\begin{pmatrix} C\dot{T} + \nabla\cdot q - Q \\ \cancel{\tau\dot{q}} + k\nabla T + q\end{pmatrix} dv + \int_{\partial e}\begin{bmatrix}\hat{T}\\ \hat{q}\end{bmatrix}\begin{bmatrix}\left((CT)^* - (CT)\right)n_t + (q^* - q)n_x \\ \cancel{\left((\tau q)^* - (\tau q)\right)n_t} + (kT^* - kT)n_x\end{bmatrix} ds = 0$$

$$\underbrace{\int_e \hat{T}(C\dot{T} + \nabla\cdot q - Q)\,dv + \int_{\partial e} \hat{T}\left((CT^* - CT)n_t + (q^* - q)n_{x=1}\right)}_{\text{base parabolic WRS}}$$

$$+ \int \hat{q}(k\nabla T + q)\,dv + \int \hat{q}(kT^* - kT)n_x = 0$$

base parabolic ...

$$+\int_e \hat{q}\,(K\nabla T + q)\,dv \qquad + \int_{\partial e} \hat{q}\,(KT^* - KT)\,n_x = 0$$

Constitutive eqn (compability between T & q)

(E)

the term we added to specify T^* on vertical faces (BC)

(5)

Two field formulation for parabolic PDE T, q are interpolated

→ 1 field formulation ($q = -K\nabla T$) & only T is interpolated

$$\int_e \tilde{T}(C\dot{T} + \nabla\cdot q - Q)\,dv + \int_{\partial e} \hat{T}\left((CT^* - CT)\,n_t + (q_n^* - q\cdot n_x)\right) ds + \int \hat{q}\,(KT^* - KT)\,n_x\,ds = 0 \qquad (6)$$

Parabolic

This reduction from a hyperbolic PDE is another approach to get the additional weight terms that multiply (T* - T). See the formulation of parabolic PDE from the last time for comparison with this approach.

Another major difference between hyperbolic and parabolic formulations is the way we specify fluxes:

Hyperbolic:
Riemann solutions

Parabolic

$$\boxed{\begin{aligned} T^* &= \{T\} + \beta[T] \\ q^* &= \{q\} - \gamma[q] \end{aligned}} + \bar{\alpha}[T]$$

not needed for parabolic PDES

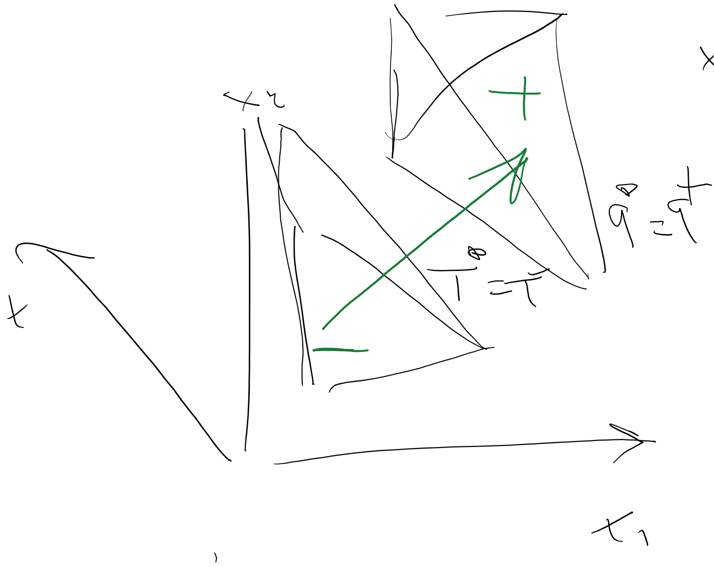
$\beta, \gamma = 0$

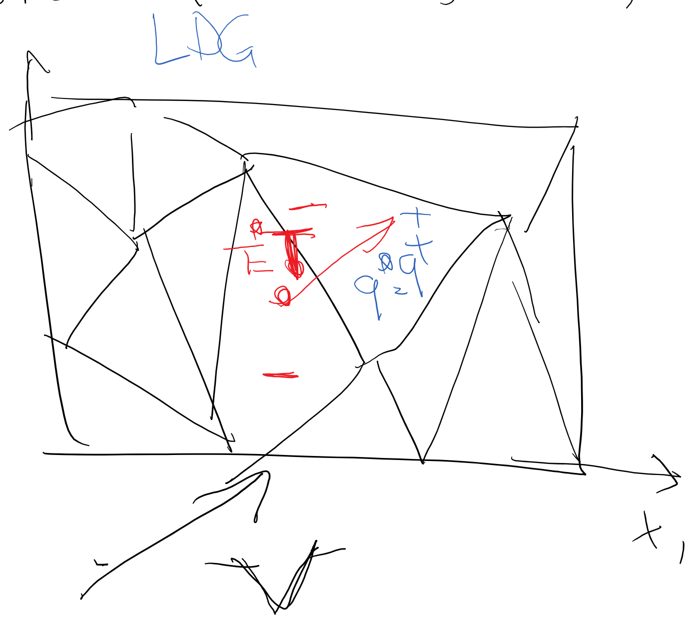
$$T^* = \{T\}$$
$$q^* = \{q\}$$

"upstream" (alternating flux)

LDG

upstream (alternating flux)





Interpolation of the solution:

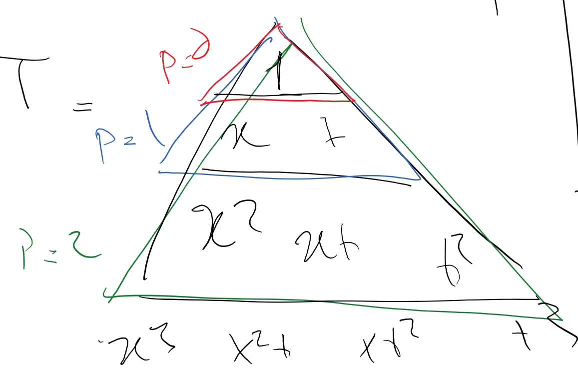
a) 2-field hyperbolic MCV equation (T and q are interpolated) 4
b) 2-field parabolic equation (T and q are interpolated) 5
c) 1-field parabolic equation (T interpolated) 6

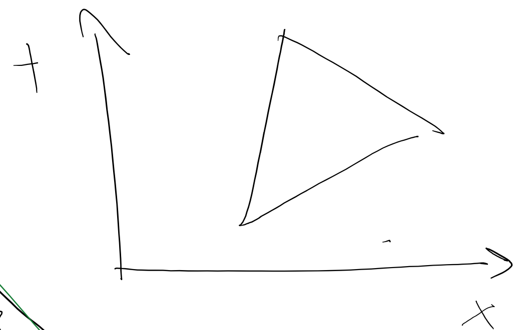
$p=1$

$T(x,t) = 1 \cdot a_1 + x a_2 + t a_3$

$\rightarrow Ka = F$

$T =$

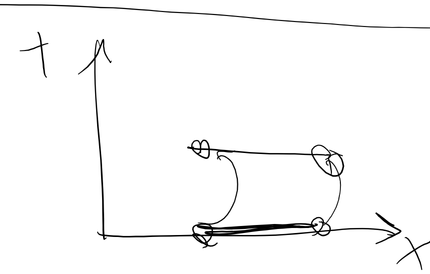




Space DG

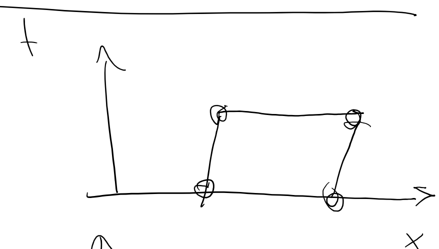
$T = a_1(t) \cdot 1 + a_2(t)\, x$

$\rightarrow \quad C\dot{a} + Ka = F$



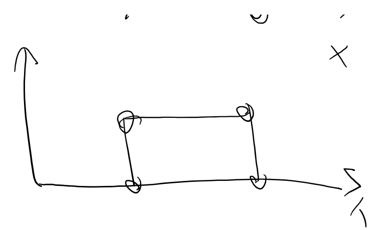
Spacetime DG
with tensorial elements

$\begin{bmatrix} 1 \\ x \end{bmatrix} \times \begin{bmatrix} 1 \\ t \end{bmatrix}$



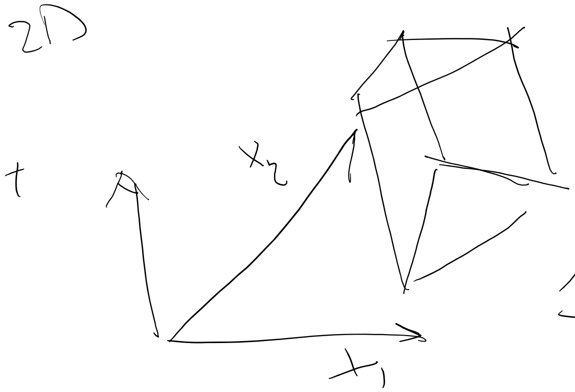
$1a_1 + x a_2 + t a_3 + xt a_4$

$$1a_1 + x a_2 + t a_3 + x t a_4$$

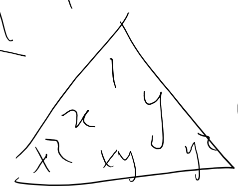


$$\longrightarrow \underline{Ka = F}$$

2D

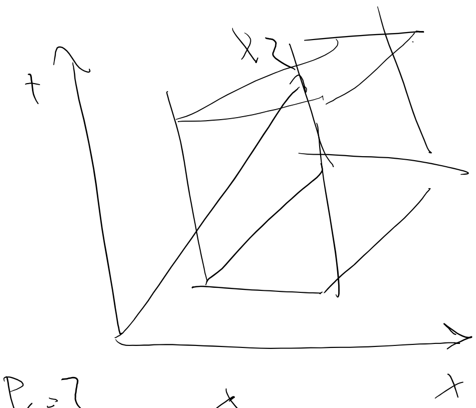


p=2 in space
p=1 in time

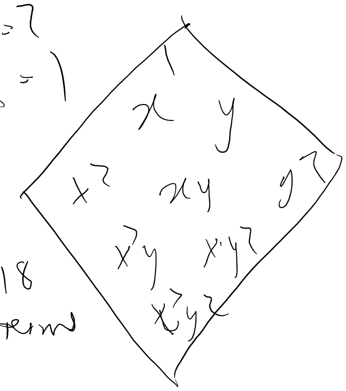


$\otimes \begin{bmatrix} 1 \\ t \end{bmatrix}$

12 terms



$P_x = 2$
$P_t = 1$



$\times \begin{bmatrix} 1 \\ t \end{bmatrix}$

18 terms

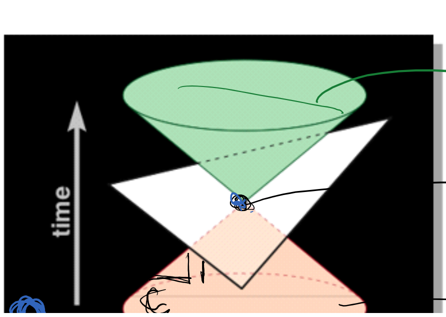
Now, we want to solve the MCV equation with cSDG (causal spacetime discontinuous Galerkin) method

$$\begin{cases} C\dot{T} + \nabla \cdot q = Q \\ \tau \dot{q} + K\nabla T = -q \end{cases}$$

wave speed $c = \sqrt{\dfrac{k}{\tau C}}$

scalar for isotropic conduction

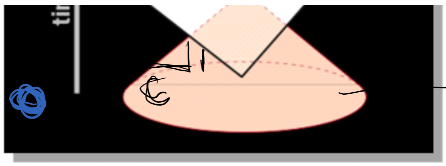
$\tau \to 0$; $C \to \infty$ (as we are tending to parabolic PDE limit where information moves infinitely fast)



domain of influence
S/n @ P only affects the soln in domain of influence

P

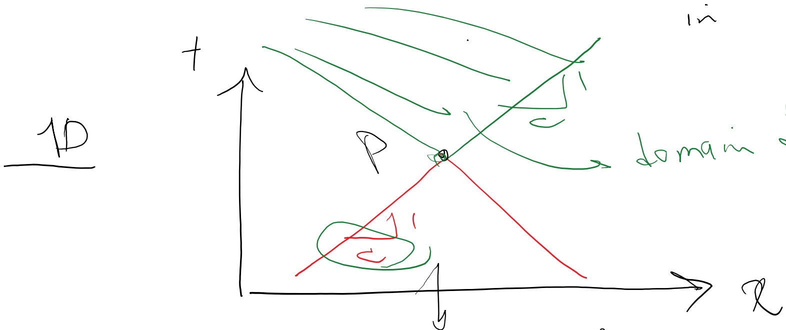
domain of dependence



causality constraint

→ domain of dependence

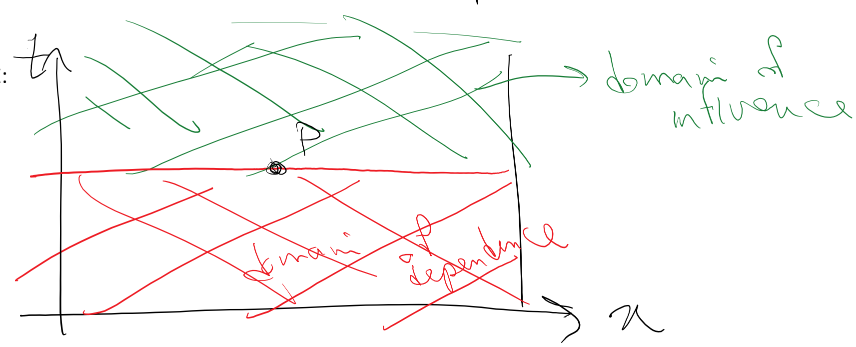
solution @ P depends on solution in domain of dependence



hyperbolic PDE

domain of dependence

For a parabolic PDE:



One problem with parabolic PDEs is that they imply information moves faster than the speed of light!

$$\tau C\ddot{T} + C\dot{T} - \nabla\cdot k\nabla T = Q + \tau\dot{Q}$$

at small length & time scales $\tau C\ddot{T}$ dominates $C\dot{T}$

hyperbolic PDE

as larger space & time scales are considered $C\dot{T}$ dominates $\tau C\ddot{T}$ & eqn can very accurately approximated by

$$\boxed{C\dot{T} - \nabla\cdot k\nabla T = Q}$$

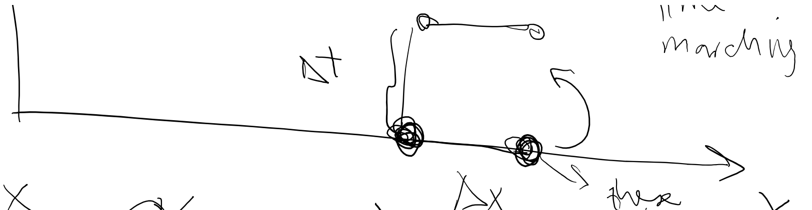
Some comments on the size of time advance for different formulations of MCV:

$c = \sqrt{\frac{k}{\tau C}}$

space DG + time

explicit time marching

space DG + time marching



$$\Delta t \le \frac{\Delta x}{c}\, Cr(p, q)$$

p → spatial order, q → temporal order

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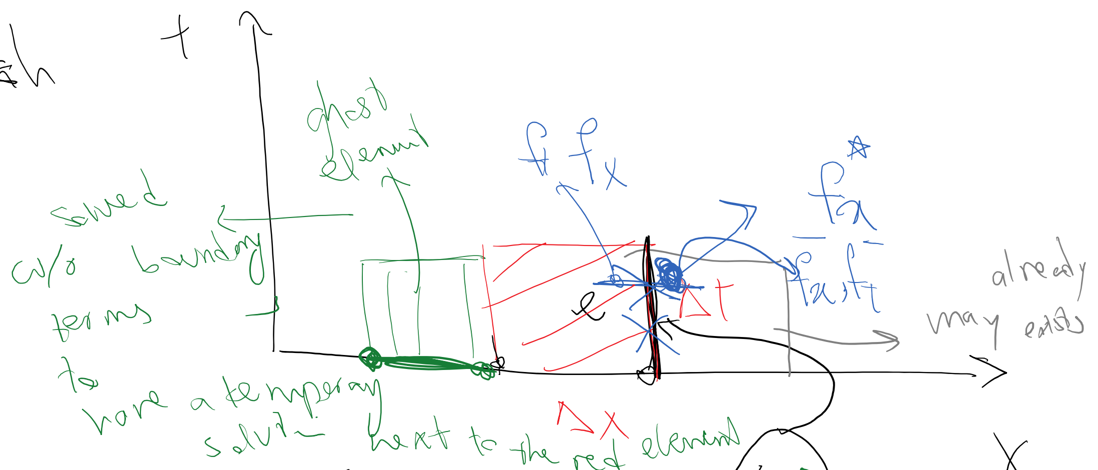
what if we use extruded spacetime meshes

(expliciat approach



$$\int_e \tilde{\omega}\,(\dot{f}_r + \nabla\cdot f_x - \rho)\,dv + \int_{\partial e} \left[\omega\,(f_r^* - f_r)\,n_t + (f_x^* - f_x)\cdot n_x\right] ds = 0$$