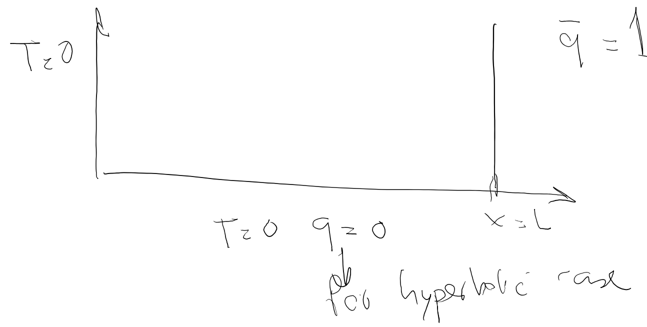
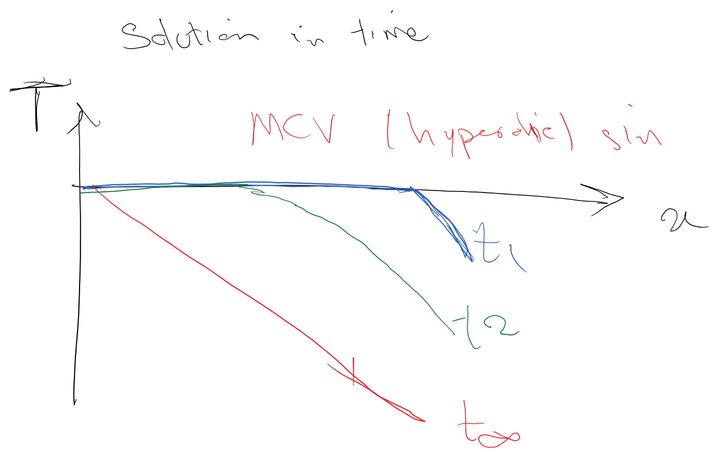
Solving the 1D heat conduction problem with causal SDG method (hyperbolic) and an explicit version of SDG method for parabolic case



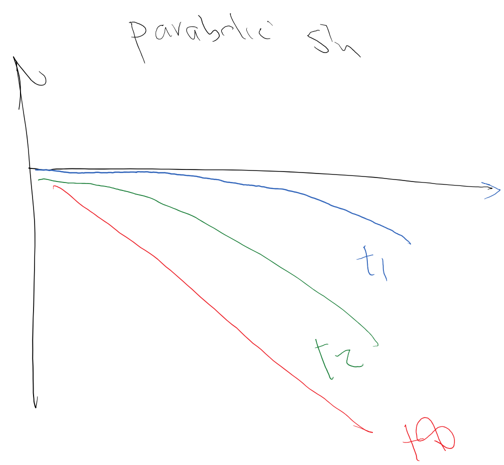
$$\begin{cases} C\dot{T} + \nabla\cdot q = Q \\ \tau\dot{q} + k\nabla T = -q \end{cases}$$

hyperbolic case

Solution in time



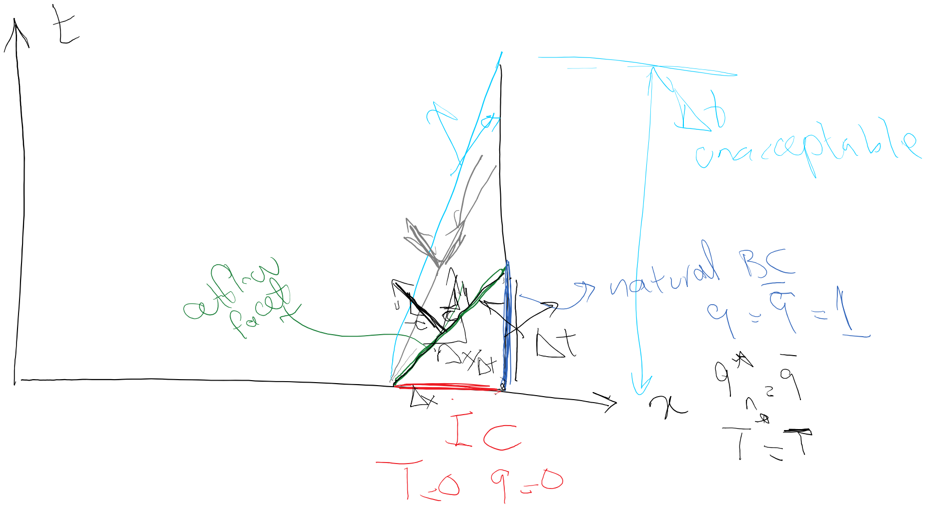
parabolic sln



For the hyperbolic case

wave speed = $c = \sqrt{\dfrac{k}{C\tau}}$

$$C\tau\ddot{T} + C\dot{T} - \nabla\cdot k\nabla T = Q + \tau\dot{Q}$$



$$\boxed{\frac{\Delta x}{\Delta t} \geq c}$$

condition

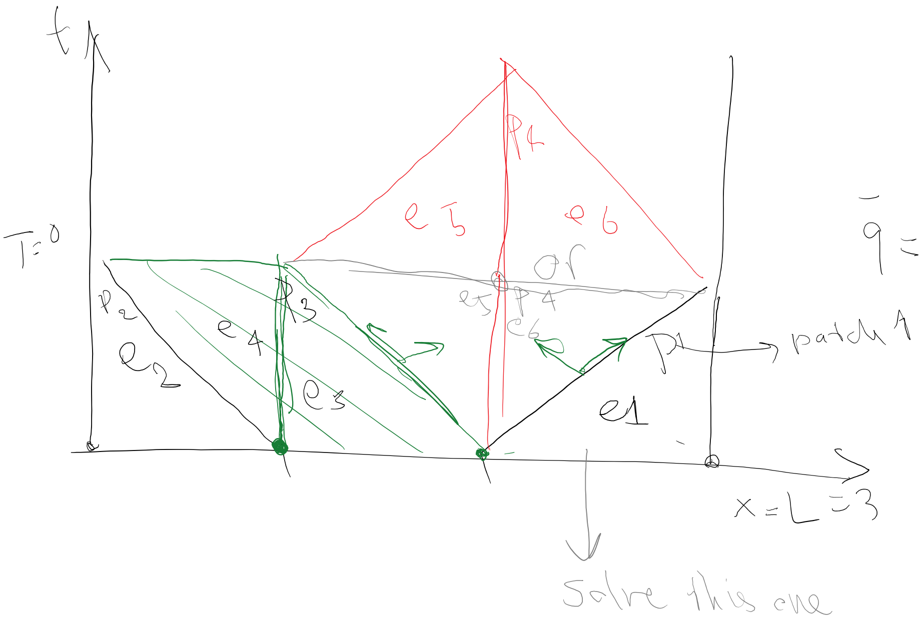
$$\boxed{\Delta t \leq \frac{\Delta x}{c}}$$

stable time advance

time advance

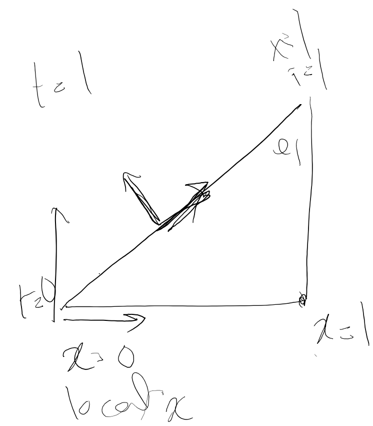
$$\frac{\Delta t}{\left(\frac{\Delta x}{c}\right)} = 1 \quad CFL = 1$$

maximum time advance of an explicit method



$$\boxed{\begin{aligned} p=1 \qquad T &= a_1 + a_2 x + a_3 t \\ q &= a_4 + a_5 x + a_6 t \end{aligned}} \quad (1)$$

$$C = 1,\ \tau = 1,\ \kappa = 1 \longrightarrow c = \sqrt{\frac{\kappa}{C\tau}} = 1$$



WRS

$$\boxed{\begin{aligned} &\int \tilde{T}(C\dot{T} + \nabla\cdot q - Q)\,dv + \int_{\partial e}\left[\tilde{T}(CT^* - CT)n_t + (q_n^* - q\cdot n_x)\right]ds \\ &+ \int \tilde{q}(\tau\dot{q} + \kappa\nabla T + q)\,dv + \int_{\partial e}\left[\tilde{q}(\tau q^* - \tau q)n_t + (\kappa T^* - \kappa T)n_x\right]ds = 0 \end{aligned}} \quad (2)$$

Discuss the boundary terms

$$I_b = \int_{\partial e} \tilde{T}(CT^* - CT)n_t + (q_n^* - q\cdot n_x)\,ds + \int_{\partial e}\tilde{q}(\tau q^* - \tau q)n_t + (\kappa T^* - \kappa T)n_x \Big]\,ds$$

$$= \int_{\partial e_i}\Big[\tilde{T}\big((1)T^*(0) - (1)T\big)(-1) + (0 - q\cdot 0)\Big]ds + \int_{\partial e_i}\tilde{q}\big((1)(0) - (1)q\big)(-1) + \big(0 - (1)T\big)(0)\,ds$$

$T^* = T \quad n_t = \frac{1}{\sqrt{2}}$

$q^* = q$

∂e_b

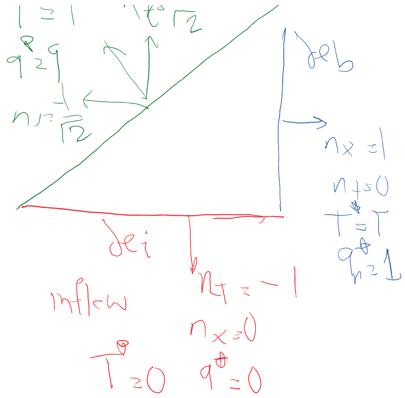
$$+\int_{\partial e_b}\hat{T}\left((1)(T)-(1)T\right)0+(1-q\cdot 1)\,ds+\int_{\partial e_b}\hat{q}\,\cancel{(1)\,q}\,(0)$$

$$\partial e_i \qquad (0)-(1)T)(1)$$

$$+\int_{\partial e_b}\left[(1)(T)-(1)T\right]n_x$$

$$+\int_{\partial e_b}\hat{T}\left(CT-CT\right)n_t+(q\cdot n-q\cdot n)\,ds\ldots$$

always zero contribution for outflow faces in WRS



$$\boxed{\text{Contributions from boundary faces are}\quad \int_{\partial e_i}(\hat{T}T+\hat{q}q)\,dx+\int_{\partial e_b}\hat{q}(1-q)\,dt}\qquad (3)$$

Interior contributions: $\qquad Q=0$

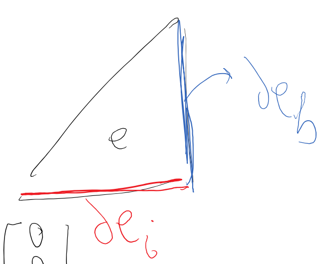
$$\int_e \hat{T}(\dot{T}+\underbrace{\nabla\cdot q}_{q_{,x}})\,dv+\int_e \hat{q}(q+T_{,x}+q)\,dv$$

$T=a_1+a_2x+a_3t \qquad T_{,x}=a_2 \qquad \dot{T}=a_3$

$q=a_4+a_5x+a_6t \qquad q_{,x}=a_5 \qquad \dot{q}=a_6$

$$\int \hat{T}(\overset{\dot{T}}{a_3}+\overset{q_{,x}}{a_5})+\hat{q}(a_6+a_2+a_4+a_5x+a_6t)\,dv \qquad (4)$$

adding all contributions $((2),(4))\Rightarrow$



$$\int_e\left[\hat{T}(a_3+a_5)+\hat{q}(a_6+a_2+a_4+a_5x+a_6t)\right]dv$$

$$+\int_{\partial e_i}\hat{T}(\overbrace{a_1+a_2x+a_3t}^{T})+\hat{q}(\overbrace{a_4+a_5x+a_6t}^{q})\,dx$$

$$\int_{\partial e_b}\hat{q}\left(1-(a_4+a_5x+a_6t)\right)dt=0 \qquad \hat{T}=\begin{bmatrix}1\\x\\t\\0\\0\\0\end{bmatrix}\quad \hat{q}=\begin{bmatrix}0\\0\\0\\1\\x\\t\end{bmatrix} \qquad (5)$$

$$\int_{\partial e_b} \hat{q}\,(1-(a_4+a_5 x+a_6 t))\,dt = 0 \qquad 1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \psi = \begin{bmatrix} 1 \\ x \\ t \end{bmatrix}$$

6 eqns & 6 unknowns ($a_1 \to a_6$)

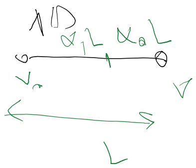
Useful formulas for integration of constant, linear, and second order polynomials inside a simplex (e.g. triangle). These ide ntities are also useful for the HW assignment

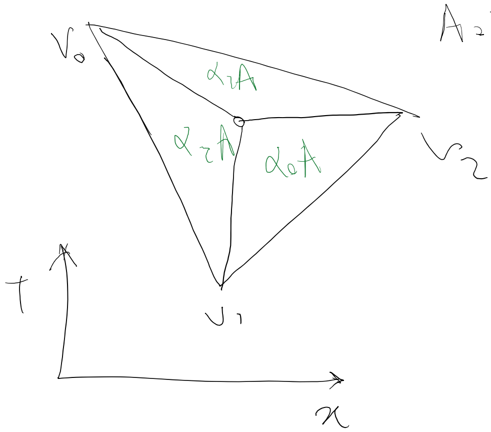
Simplices

0D

1D

$\alpha_0+\alpha_1=1$





A = area

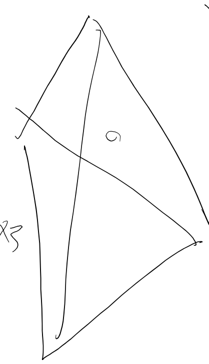
$\alpha_0+\alpha_1+\alpha_2=1$

2D

$(x,t) \longleftrightarrow (\alpha_0, \alpha_1, \alpha_2)$

(α_1, α_2)

$\alpha_0, \alpha_1, \alpha_2, \alpha_3$



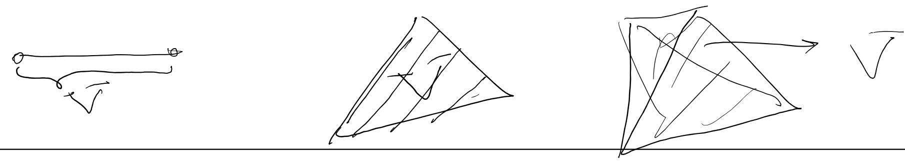
$$\sum_{i=0}^{d} \alpha_i = 1$$

1 α is redundant

often α_0 or α_d is removed.

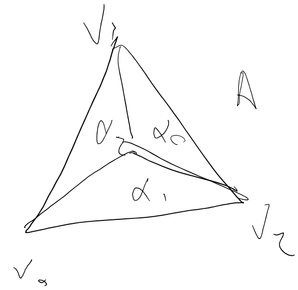
(6)
$$\int_V \alpha_0^{p_0}\,\alpha_1^{p_1}\cdots\alpha_d^{p_d}\,dV = \frac{d!\,p_0!\,p_1!\cdots p_d!}{(d+p_0+p_1+\cdots+p_d)!}\,|V|$$

$|V|$ → measure of simplex

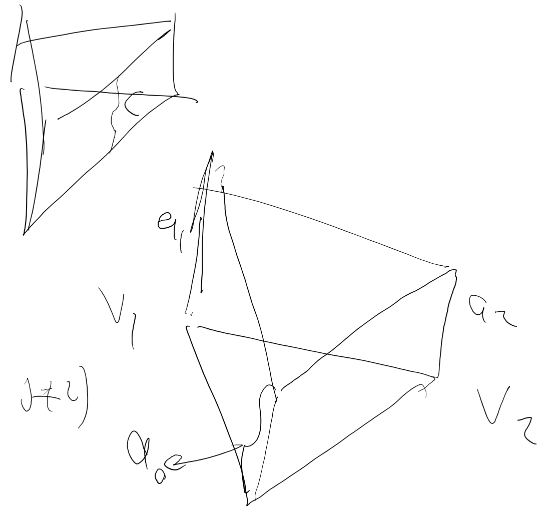


for a triangle

(6)
$$\int \alpha_0^{p_0}\,\alpha_1^{p_1}\,\alpha_2^{p_2}\,dA = \frac{2!\,p_0!\,p_1!\,p_2!}{(2+p_0+p_1+p_2)!}$$



Since our elements are order p = 1, integrands are of order 2 (maximum order of integration).
We want to find close form expressions for order 0, 1, and 2 polynomial integrals for a triangle



– 0 $\quad \int_A c\, dA = cA$

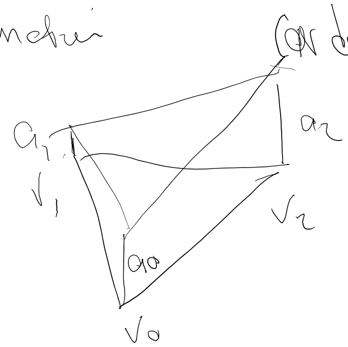
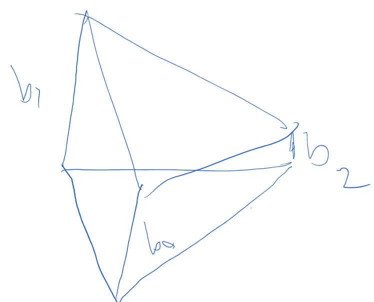
– 1 $\quad f_e(\alpha) = a_0\alpha_0 + a_1\alpha_1 + a_2\alpha_2$

$f(@v_i) = a_i \quad (\alpha_i = 1 \;\&\; \alpha_j = 0,\ j \neq i)$

$$\int_A \alpha_0\, dA = \int \alpha_0^1 \alpha_1^0 \alpha_2^0\, dA = \frac{2!\, p_0!\, p_1!\, p_2!}{(2+p_0+p_1+p_2)!} = \frac{2!\,1!\,0!\,0!}{3!} = \frac{1}{3}$$

$$\int_A f(\alpha)\, dA = \int_A [a_0\alpha_0 + a_1\alpha_1 + a_2\alpha_2]\, dA = \frac{a_0+a_1+a_2}{3}$$

– linear × linear function (order = 2)

$f_a(\alpha)$  ×  $f_b(\alpha)$

$$\int_A f_a(\alpha)\, f_b(\alpha)\, dA = \int (a_0\alpha_0 + a_1\alpha_1 + a_2\alpha_2)(b_0\alpha_0 + b_1\alpha_1 + b_2\alpha_2)\, dA =$$

$$\int_A \Big[a_0b_0\alpha_0^2 + a_1b_1\alpha_1^2 + a_2b_2\alpha_2^2 + (a_0b_1 + a_1b_0)\alpha_0\alpha_1 + (a_1b_2 + a_2b_1)\alpha_1\alpha_2 + (a_2b_0 + a_0b_2)\alpha_0\alpha_2 \Big]\, dA$$

$$\int_A \alpha_0^2\, dA = \int_A \alpha_0^2 \alpha_1^0 \alpha_2^0\, dA = \frac{2!\,(2!)(0!)(0!)}{(2+2+0+0)!} = \frac{1}{6}$$

$$\int_A \alpha_0\alpha_1\, dA = \int_A \alpha_0^1 \alpha_1^1 \alpha_2^0\, dA = \frac{2!\,(1!)(1!)(0!)}{(2+1+1)!} = \frac{1}{12}$$

$$A \qquad A \qquad \overline{(2+1+1)} = T_2$$

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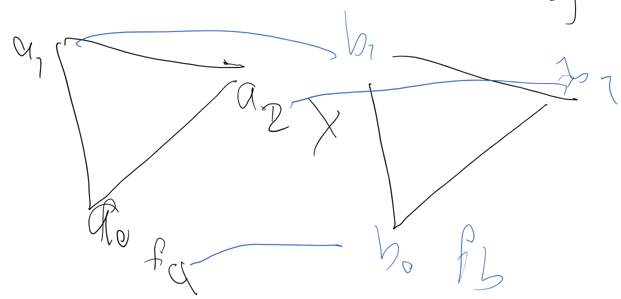
Summary

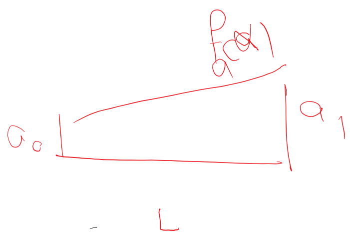
$$\int_A f_a(\alpha) f_b(\alpha)\, dA = \left\{\frac{1}{6}\left[a_0b_0 + a_1b_1 + a_2b_2\right] + \frac{1}{12}\left[a_0b_1 + a_1b_0 + a_1b_2 + a_2b_1 + a_2b_0 + a_0b_2\right]\right\} A$$

$$\int f_a(\alpha)\, dA = \left(\frac{a_0 + a_1 + a_2}{3}\right) A$$

$$\int c\, dA = cA$$



Similar integrals for lines are:



2nd order

$$\int_L f_a(\alpha) f_b(\alpha)\, dl = \left[\frac{a_0b_0 + a_1b_1}{3} + \frac{a_0b_1 + a_1b_0}{6}\right] L$$

$$\int_L f_a(\alpha)\, dl = \frac{a_0 + a_1}{2} L$$

$$\int_L c\, dl = cL$$

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