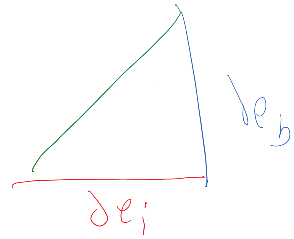
DG2020/04/01

Wednesday, April 1, 2020 11:41 AM

Weighted residual copied for the element on the right boundary

$$\iint_e \left[\hat{T}(a_3+a_5) + \hat{q}(a_6+a_2+a_4+a_5 x + a_6 t)\right] dv$$



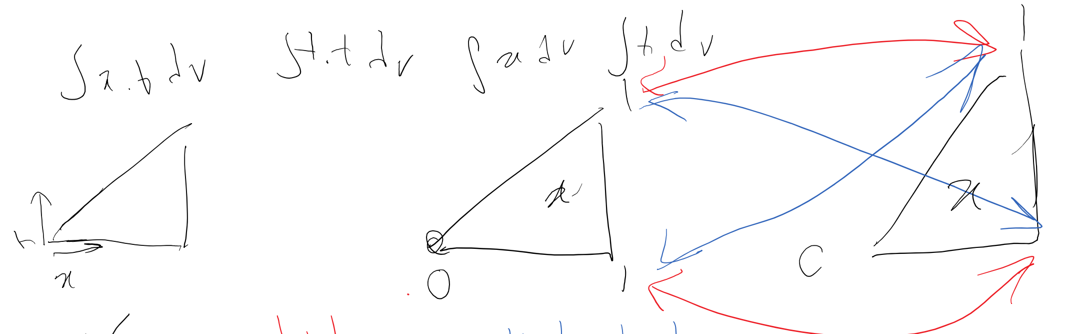
$$+ \int_{\partial e_i} \hat{T}(a_1+a_2 x+a_3 t) + \hat{q}(a_4+a_5 x + a_6 t)\, dt$$

$$+ \int_{\partial e_b} \bar{q}\,(1-(a_4+a_5 x + a_6 t))\, dt = 0$$

$$\hat{T} = \begin{bmatrix} 1 \\ x \\ t \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \hat{q} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ x \\ t \end{bmatrix}$$

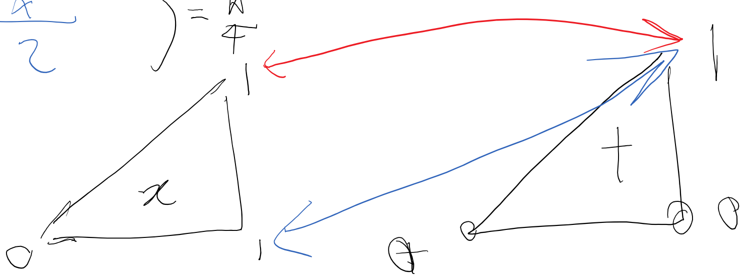
$\int x \cdot x\, dv \quad \int x \cdot t\, dv \quad \int t \cdot t\, dv \quad \int x\, dv \quad \int t\, dv$

$\int_e x \cdot x\, dv$



$$\int_e x \cdot x\, dv = A\left(\frac{1 \times 1 + 1 \times 1}{6} + \frac{1 \times 1 + 1 \times 1}{12}\right) = \frac{A}{2} \quad \left(A = \frac{1}{2}\right)$$

$$\int x \cdot t = A\left(\frac{1 \times 1}{6} + \frac{1 \times 1}{12}\right) = \frac{A}{4}$$



all interior integrals

$$\int_e x\, dA = \frac{2}{3} A \qquad \int_e t\, dA = \frac{1}{3} A$$

$$\int_e x^2\, dA = \frac{1}{2} A \qquad \int_e t^2\, dA = \frac{1}{6} A \quad ★$$

$$\int_e x t\, dA = \frac{1}{4} A$$

$$I_i = \iint \left(\begin{bmatrix} 1 \\ x \\ t \\ 0 \\ 0 \\ 0 \end{bmatrix} (a_3+a_5) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ x \\ t \end{bmatrix} (a_6 + \underbrace{a_2 + a_4} + a_5 x + a_6 t) \right) dA$$

interior integral

$$I_I \text{ interior integral} = \int\!\!\int \begin{bmatrix} 1 \\ t \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} (a_3 + a_5 \ldots) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ x \\ t \end{bmatrix} (a_6 + a_2 + a_4 + a_5 x + \ldots) \ldots$$

$$= A \left\{ \begin{bmatrix} a_3 + a_5 \\ \frac{2}{3}(a_3 + a_5) \\ \frac{1}{3}(a_3 + a_5) \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_6 + a_2 + a_4 + a_5 \times \frac{2}{3} + a_6 \times \frac{1}{3} \\ (a_6 + a_2 + a_4) \times \frac{2}{3} + a_5 \frac{1}{2} + a_6 \frac{1}{4} \\ (a_6 + a_2 + a_4) \frac{1}{3} + a_5 \times \frac{1}{4} + a_6 \times \frac{1}{6} \end{bmatrix} \right\}$$

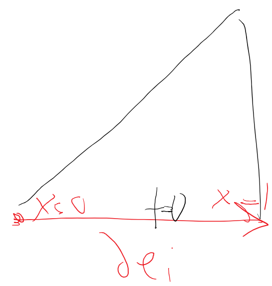
use ⊗

1a

Inflow face integral:

$$I_{inf} = \int_{\partial e_{i,1}} \hat{T}\left(a_1 + a_2 x + \cancel{a_3 t}\right) + \hat{q}\left(a_4 + a_5 x + \cancel{a_6 t}\right) ds \qquad t=0 \qquad ds \to dx$$

$$\int_{x=0}$$



We can either use constant, linear, and bi-linear integrals for a line from the last session or in this particular case, simply integrate over x

$$\int_0^1 \begin{bmatrix} 1 \\ x \\ \cancel{x} \\ 0 \\ 0 \\ 0 \end{bmatrix} (a_1 + a_2 x + \cancel{a_3 t}) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ x \\ \cancel{t} \end{bmatrix} (a_4 + a_5 x + \underbrace{a_6 t}_{0}) \, dx$$

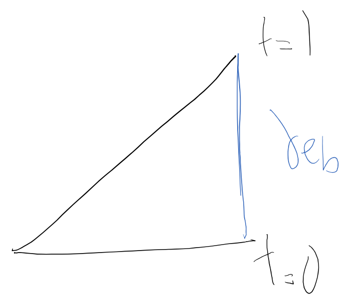
$$\boxed{I_{inf} = \begin{bmatrix} a_1 + a_2/2 \\ a_1/2 + a_2/3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_4 + a_5/2 \\ a_4/2 + a_5/3 \\ 0 \end{bmatrix}}$$

1b

Finally, we have the boundary integral

$$I_b = \int_{\partial e_t} \hat{q}\left(1 - a_4 - a_5 x - a_6 t\right) dt$$

$$= \int \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ x \\ t \end{bmatrix} \left(1 - a_4 - a_5(1) - a_6(t)\right) dt \qquad x \to 1$$



$$J_{jb} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 - a_4 - a_5 - a_6/2 \\ 1 - a_4 - a_5 - a_6/2 \\ 1/2 - a_4/2 - a_5/2 - a_6/3 \end{bmatrix}$$

1c

go to the RHS

1a, 1b, 1c

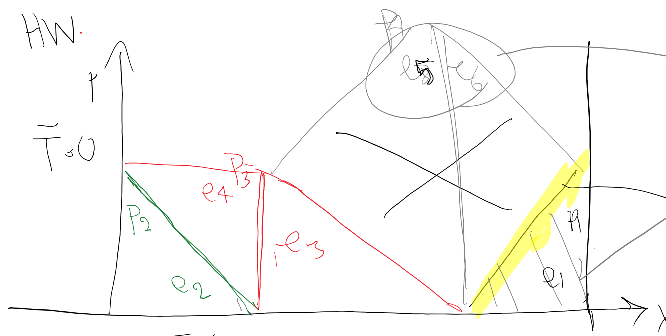
$I_{Int} + I_{jb} + J_{jb} = 0 \Longrightarrow$

$$K_{6\times 6}\, a_{6\times 1} = F_{6\times 1} \longrightarrow a = \begin{bmatrix} 0 \\ 0 \\ -3.27 \\ -6 \\ 3.27 \\ 15.27 \end{bmatrix}$$

$$T = -3.27t$$

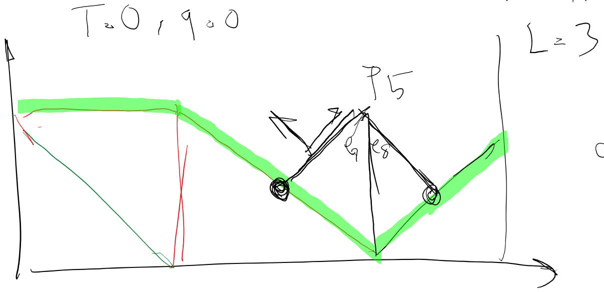
$$q = -6 + 3.27x + 15.27t$$

HW.



errors are large

because of this face the solution of p4 is nonzero

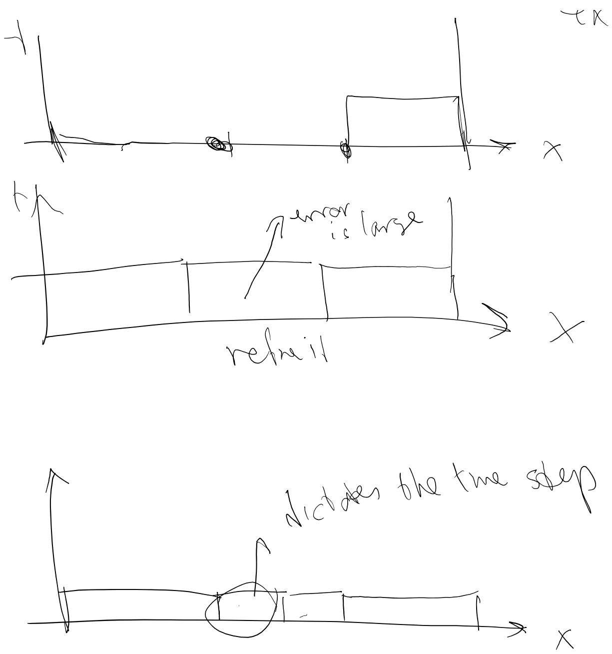


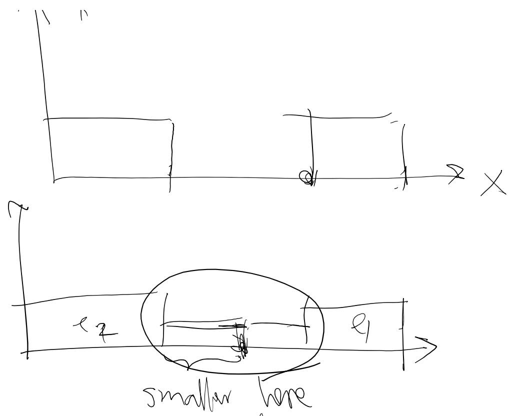
after the refinement front below the rejected element gets refined

Because of the refinement both spatial and temporal size of the element decrease.

Comparison with a time marching scheme

explicit



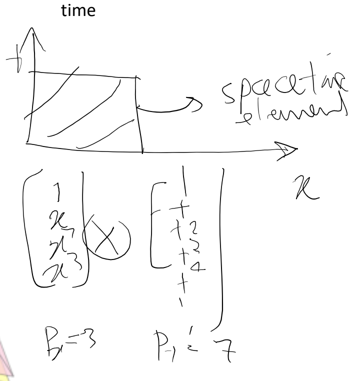


the time step of e_1 & e_2 is now invalid

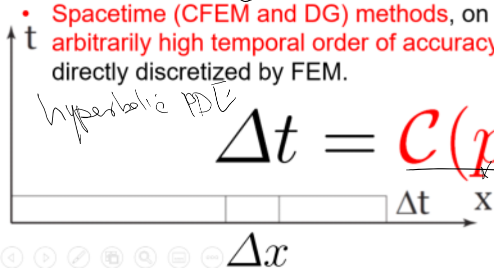
Advantages of cSDG method:

- Achieving high temporal orders in semi-discrete methods (CFEMs and DGs) is very challenging as the solution is only given at discrete times.
- Perhaps the most successful method for achieving high order of accuracy in semi-discrete methods is the Taylor series of solution in time and subsequent use of Cauchy-Kovalewski or Lax-Wendroff procedure (FEM space derivatives ⇨ time derivatives). However, this method becomes increasingly challenging particularly for nonlinear problems.
- High temporal order adversely affect stable time step size for explicit DG methods (e.g. $\frac{1}{2p+1}$ or worse for RKDG and ADER-DG methods).

Achieving high temporal order in time marching methods is difficult
- Any spacetime method can easily achieve high order of accuracy in time



- Spacetime (CFEM and DG) methods, on the other hand can achieve arbitrarily high temporal order of accuracy as the solution in time is directly discretized by FEM.

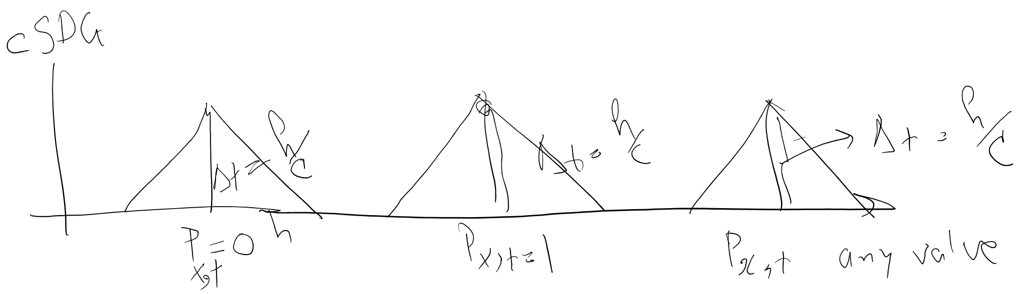


$$\Delta t = C(p)\,\Delta x / c$$

C depends on spatial & temporal discretization types & orders in spacetime

$P_t = 5 \quad C \sim \frac{1}{11} \approx 0.1$

For a 5th order method in time, the maximum time advance is reduced by about an order of magnitude.



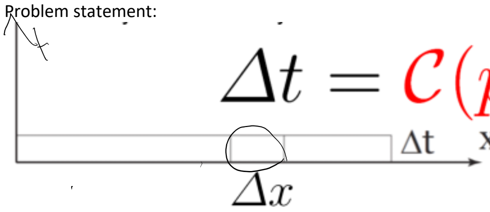
In cSDG method, since the only constraint is keeping the outflow facets causal, the order of element has no effect on the maximum time advance of a vertex.

- Just because of this simple fact, for p = 5, the maximum step of cSDG method is about 10x of EXRK5!

1. Asynchronous / no global time step

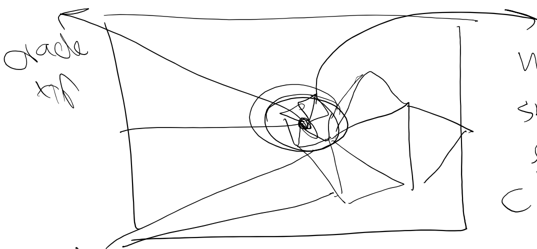
Problem statement:

$$\Delta t = C(p)\Delta x / c$$

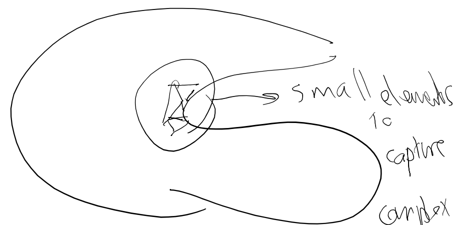


explicit method
smallest element dictates global time step

reasons for small elements

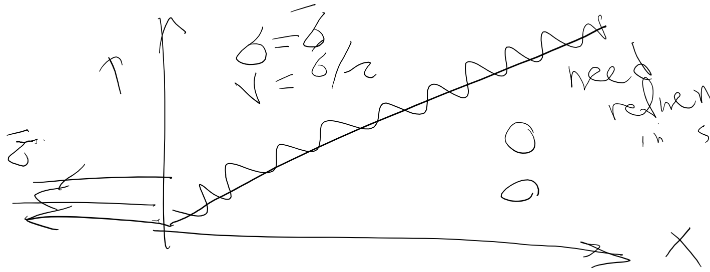


need small elements close to high gradient solution features



small elements to capture complex geometries

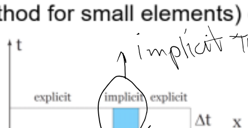
fluids need refined elements



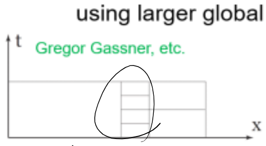
need refinement in spacetime regions of small elements can change in spacetime

- **Implicit-Explicit (IMEX)** methods increase the time step by geometry splitting (implicit method for small elements) or operator splitting.

implicit time integration for small elements



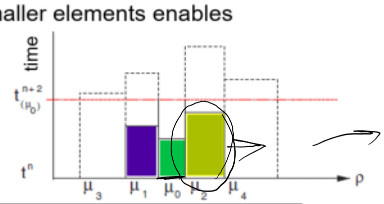
- **Local time-stepping (LTS)**: subcycling for smaller elements enables using larger global time steps



subcycling

A. Taube, M. Dumbser, C.D. Munz and R. Schneider, A high-order discontinuous Galerkin method with time accurate local time stepping for the Maxwell equations, Int. J. Numer. Model. 2009; 22:77–103

ADER DG



local time stepping (LTs

With cSDG method

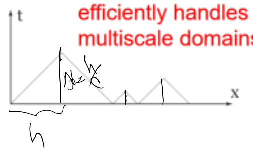
aSDG

- Small elements locally have smaller progress in time (no global time step constrains)
- None of the complicated "improvements" of time

SDGFEM graciously and efficiently handles highly multiscale domains aSDG

- Small elements locally have smaller progress in time (no global time step constrains)
- None of the complicated "improvements" of time marching methods needed

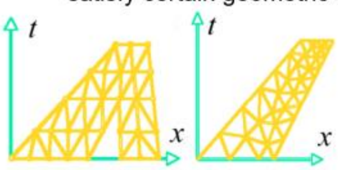
SDGFEM graciously and efficiently handles highly multiscale domains



Each element takes its own 100% efficient maximum time advance.

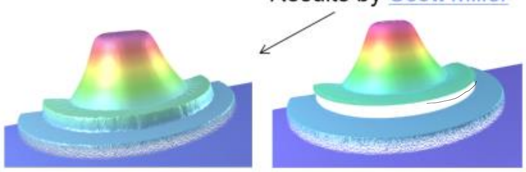
2. Spacetime grids and Moving interfaces

- Problems with moving interfaces:
  * Solid-fluid interaction * Non-linear free surface water waves
  * Helicopter rotors /forward fight * Flaps and slats on wings and piston engines
- Derivation of a conservative scheme is very challenging:
  - Even Arbitrary Lagrangian Eulerian (ALE) methods do not automatically satisfy certain geometric conservation laws.

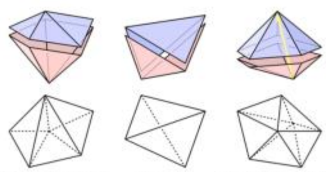


Tracking moving boundaries in spacetime

Results by Scott Miller

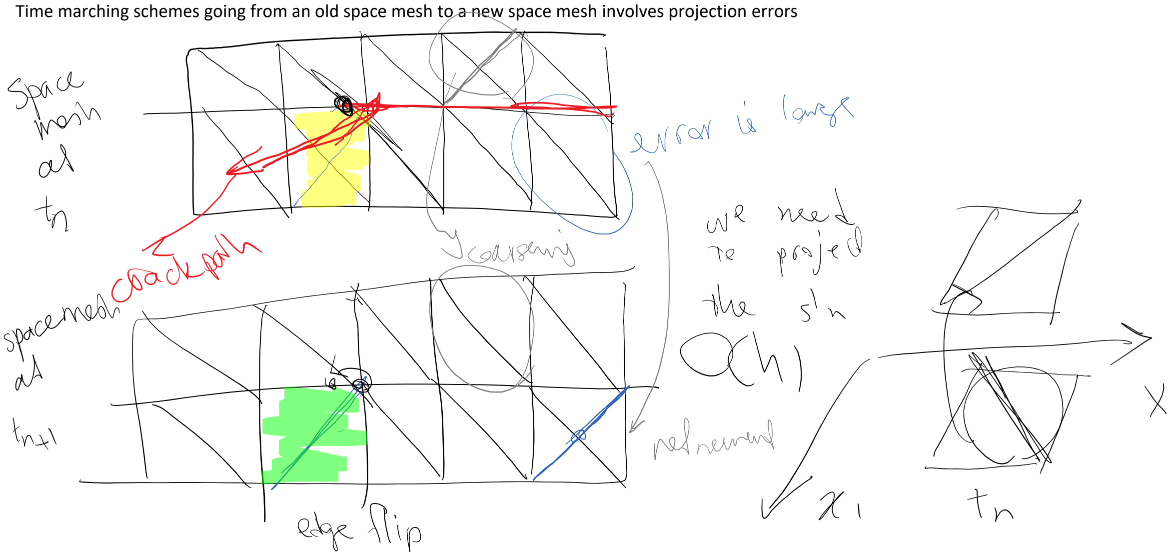


- Spacetime mesh adaptive operations
  Enable mesh smoothing and adaptive operations Without projection errors of semi-discrete methods.

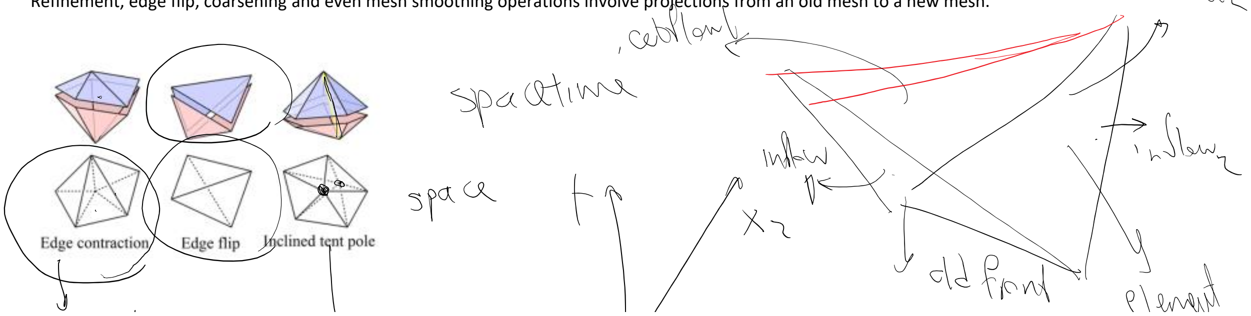


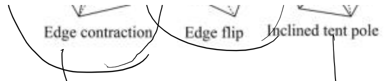
Edge contraction Edge flip Inclined tent pole

Time marching schemes going from an old space mesh to a new space mesh involves projection errors

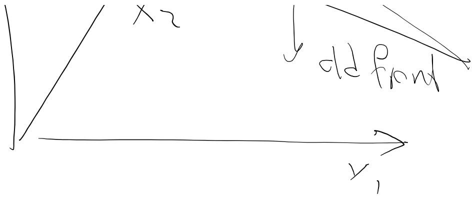


Refinement, edge flip, coarsening and even mesh smoothing operations involve projections from an old mesh to a new mesh.

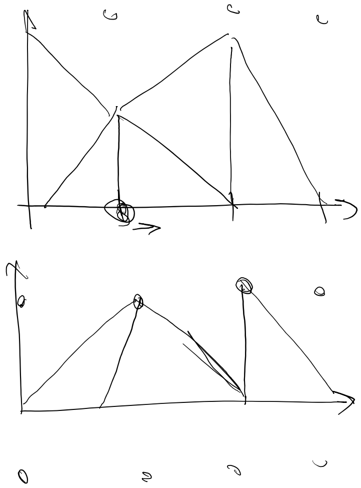




Coarsening
5 elements → 3 elements



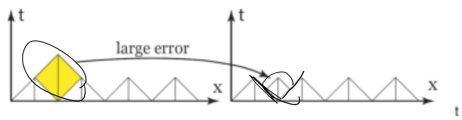
element in space time



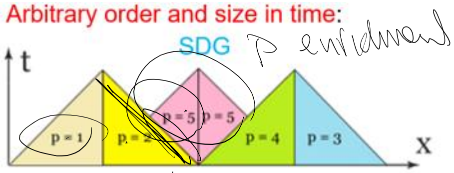
tilted tent pole is used for smoothing

3. Adaptive mesh operations

- Local-effect adaptivity: no need for reanalysis of the entire domain



- Arbitrary order and size in time:



p enrichment

h-refinement

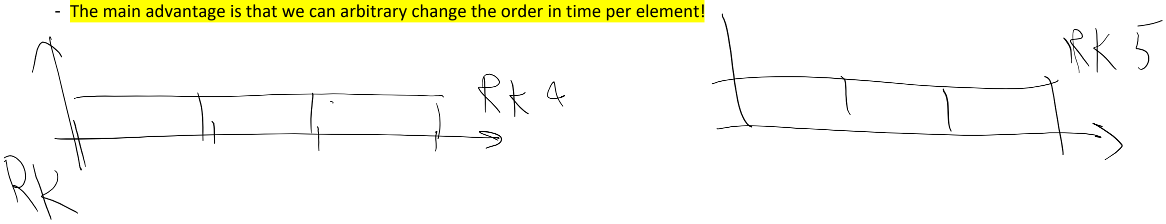
Example:
LTS by Dumbser, Munz, Toro, Lorcher, et. al.

ADER-DG

order in time can change element by element

h-refinement and p-enrichment can easily be done as with all the other DG methods (compared to CFEMs)

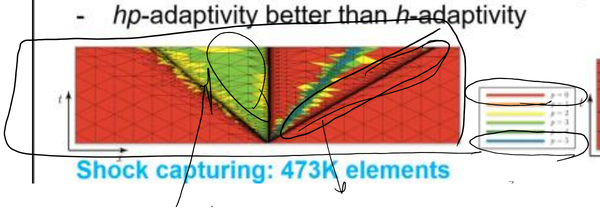
- The main advantage is that we can arbitrary change the order in time per element!



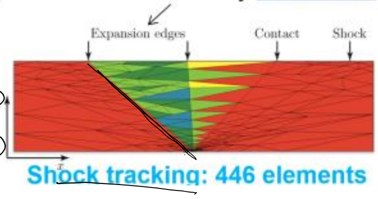
With time marching schemes it is very difficult to have different temporal orders for different parts of the domain.

- Adaptive operations in spacetime:
  - Front-tracking better than shock capturing
  - *hp*-adaptivity better than *h*-adaptivity

Sod's shock tube problem
Results by Scott Miller



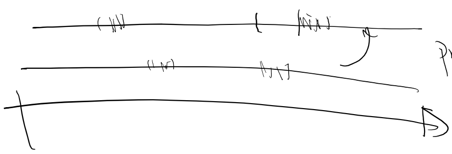
Shock capturing: 473K elements



Shock tracking: 446 elements

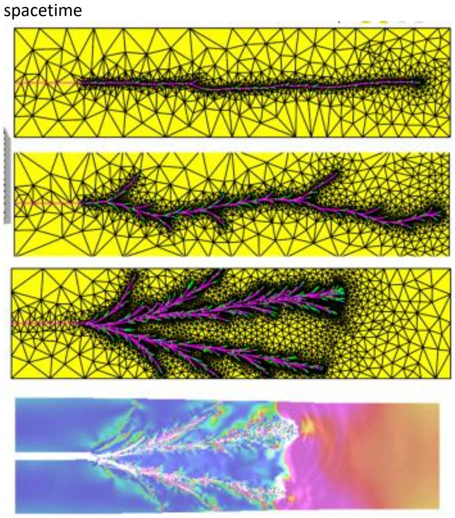
high order elements

p = 0 along the shock

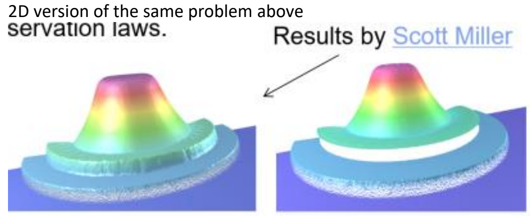


projection errors

Same shock tracking schemes can be used to track crack faces in spacetime



2D version of the same problem above



shock capturing

shock tracking discontinuity