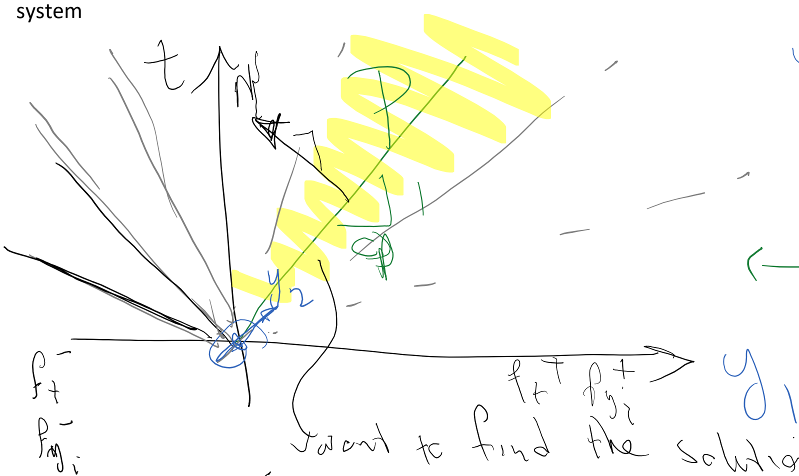
DG2020/04/08

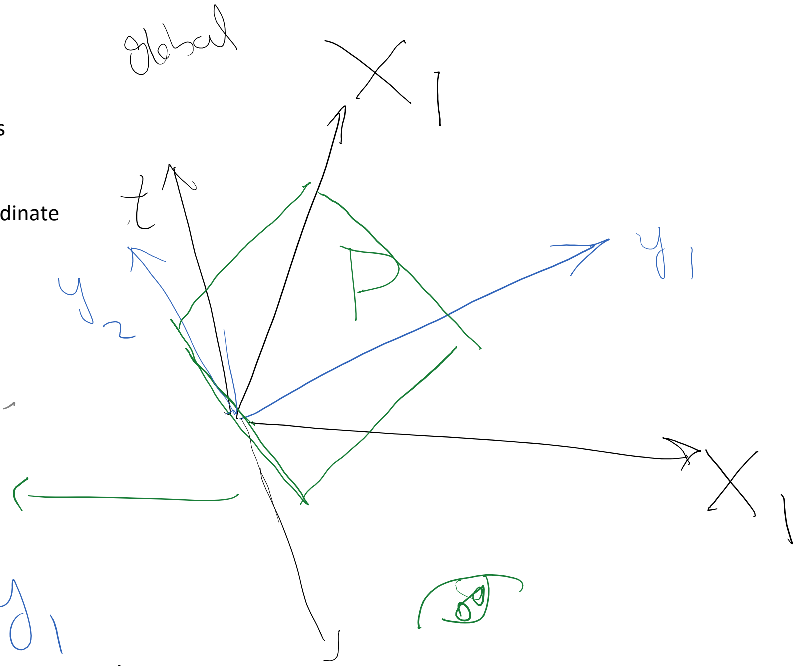
Wednesday, April 8, 2020 11:39 AM

I referred you to 2018 notes for linear Riemann solutions for hyperbolic PDEs

Today, we solve a 2D problem.
In 2D and 3D problems it's easier to solve the Riemann problem in local coordinate system



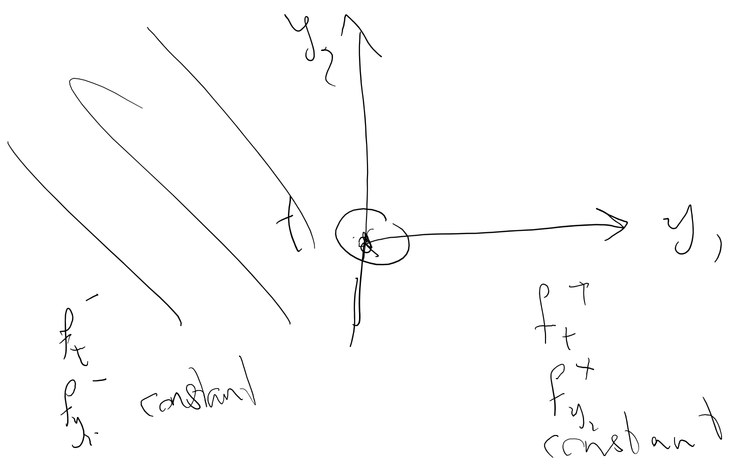
I want to find the solution here



$$N_{yt} = \begin{bmatrix} -1/\sqrt{cp^2+1} \\ 0 \\ \frac{cp}{\sqrt{1+cp^2}} \end{bmatrix} \begin{array}{l} \longrightarrow n_{y_1} \\ \longrightarrow \boxed{n_{y_2} = 0} \\ \\ \end{array}$$

y_1, y_2 are local axes

y is set up such that only $n_{y_1} \neq 0$



things change only along y_1

$$\dot{\rho}_t + \nabla_y \cdot F_y = S$$

$$\dot{\rho}_t + \nabla_y \cdot [f_{y_1}\ f_{y_2}\ f_{y_3}] = S$$

$$\dot{\rho}_t + \underbrace{f_{y_1,y_1}}_{\text{1D problem}} + \underbrace{\underbrace{\cancel{f_{y_2,y_2}}}_{0} + \underbrace{\cancel{f_{y_3,y_3}}}_{0}}_{\text{not even computed}} = S$$

That's the motivation of going to a local coordinate system, to get rid of derivatives with respect to y2 (and y3)

Fy is obtained from Fx using coordinate transformation rules.

Solid Mechanics

$$\dot{p} + \nabla \cdot \sigma = 0$$

↓ spatial flux
2nd order tensor

$\sigma_y = Q \sigma_x Q^T$

Another thing

on facets we need

$$F^* \cdot N = f_t^* \cdot n_t + f_y^* \cdot n_y$$
$$= f_t^* n_t + f_{y_1}^* n_{y_1} + \cancel{f_{y_2}^* n_{y_2}}$$

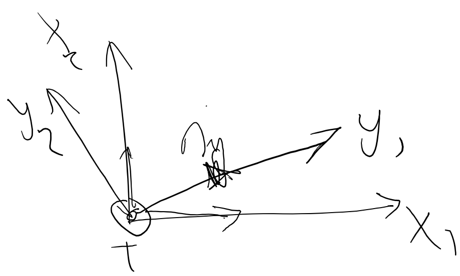
$$N_{yt} = \begin{bmatrix} -1/\sqrt{cp^2+1} \\ 0 \\ \frac{cp}{\sqrt{1+cp^2}} \end{bmatrix} \begin{matrix} \rightarrow n_{y_1} \\ \rightarrow \boxed{n_{y_2} = 0} \\ \\ \end{matrix}$$

By going to local coordinate system, not only we directly solve a 1D Riemann solution, but we don't end up computing things that we don't even need (f*y2 is not needed and we won't compute it)

There is another way to formulate Riemann solution

$$\dot{q} + A_1^X q_{,x_1} + A_2^X q_{,x_2} + A_3^X q_{,x_3} = S$$

$$q_{,x} = [A_1^X q, A_2^X q, A_3 q]$$



coordinate transformation

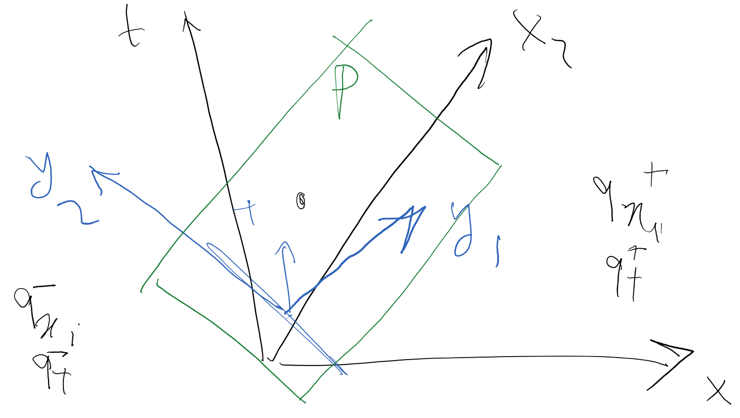
$$q_{,y_1} = A_1^X n_{y_1} + A_2^X n_{y_2} + A_3^X n_{y_3} = A_n$$

$$\boxed{\begin{matrix} q_{,y_2} = 0 \\ q_{,y_3} = 0 \end{matrix}}$$

because of Riemann problem set-up

$$\rightarrow \quad \dot{q} + A_n q_{,y_1} = S$$

① $q^-_{x_i}$ & $q^+_{x_i}$ are given in global coordinate system



② use $Q = \begin{bmatrix} n_{y_1} \\ n_{y_2} \\ n_{y_3} \end{bmatrix}$

to compute

$q^-_{y_i}$ & $q^+_{y_i}$

③ solve the Riemann problem in local coordinate system

④ If needed go back to the original global coordinate system. (only needed if weak statement on the facet is in global coord.) not recommended.

(*) In general the WRS (WK) are objective, meaning that they have the same tensorial expression in x and y. So, it's much easier to directly write the WK in y (local) coordinate system to begin with.

example

Elastodynamic weak statement:

stress

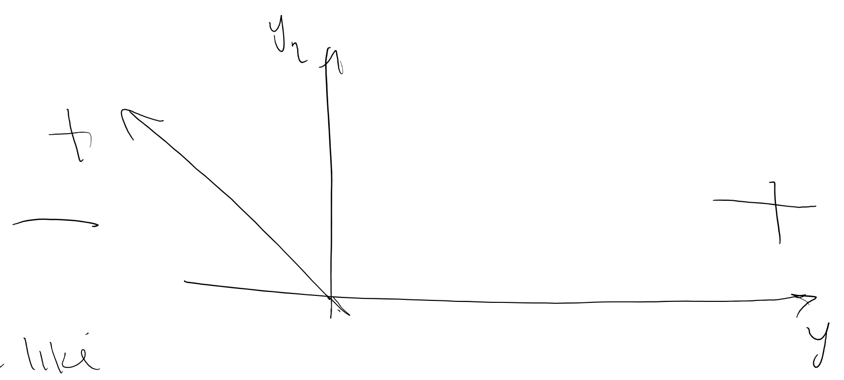
$$\int_Q \{-\dot{w}_{i,j}\sigma_h^{ij} + \rho\ddot{w}_i\dot{u}_h^i + \rho\dot{w}_i b^i\}\mathrm{dV} + \underbrace{\int_{\partial Q}\{\dot{w}_i(\sigma_h^{ij})^* n_j - \dot{w}_i(p_h^i)^* n_i - [\![E_{ij}]\!]\hat{\sigma}^{ij}n_j + [\![\dot{u}_i]\!]\hat{\sigma}^{ij}n_j\}\mathrm{dS}}_{\text{Strain}}$$

$$+ \int_{\partial Q^u} (w_0)_i [\![u^i]\!] n_i \mathrm{dS} = 0 \quad \forall \mathbf{w} \in V^Q, \qquad \text{(A.1)}$$

E_{ij} σ_{ij} can be expressed in x or y systems

3D elastodynaimcss problem

y_3 is in plane of y_2 & t



kinematic

force like

$r_{3\times1}$: v

$v_{3\times1} = \dot{u}_{3\times1}$

$\epsilon_{3\times3} = \tilde{\nabla} u_{3\times1}$

$\sigma_{3\times3} = C_{3\times3\times3\times3}\,\epsilon_{3\times3}$

Voigt notation

$$\gamma = \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{12} \\ 2\epsilon_{23} \\ 2\epsilon_{31} \end{bmatrix} \qquad S = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix}$$

$S_{6\times1} = \bar{C}_{6\times6}\,\gamma_{6\times1}$ Voigt stiffness

$\gamma_{6\times1} = \bar{D}_{6\times6}\,S_{6\times1}$ Voigt compliance

balance of lin. momentum

$\dot{p} - \nabla\cdot\sigma = s$ ($\tilde{\nabla} S$)

compatibility $\dot{\epsilon} - \tilde{\nabla} v = 0$

balance law need ( )

stress & v appear as spatial fluxes

i,j $\quad \frac{1}{2}(u_{i,j} + u_{j,i}) -$

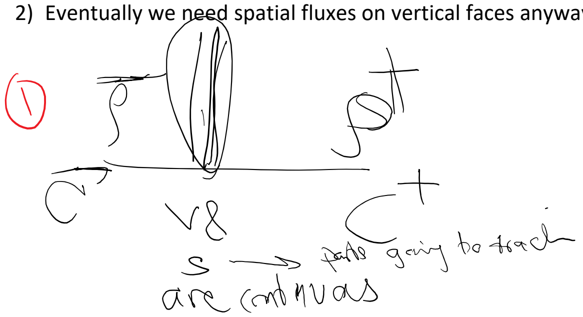
$\boxed{\epsilon - \tilde{\nabla} u = 0}$ why not this $\quad \frac{1}{2}(v_{i,j} + v_{j,i}) = 0$

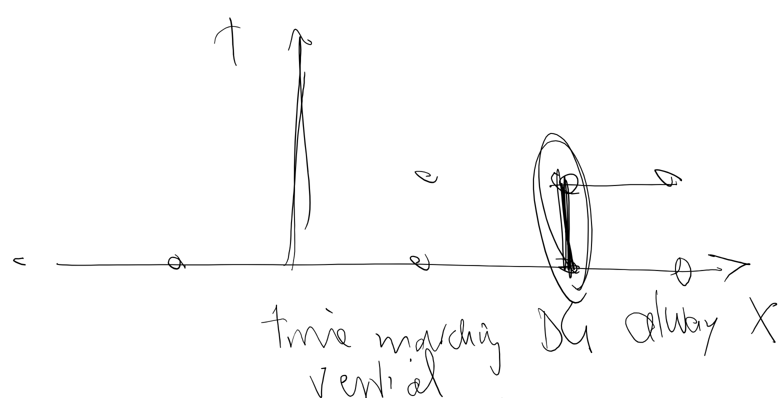
Kinematic: v, γ

force-like: $p = \rho v$, $S = \bar{C}\gamma$

I prefer to work with spatial flux quantities in vector q because:

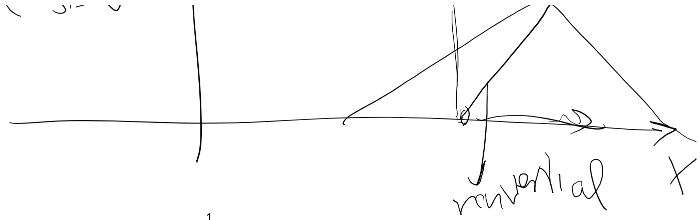
1) In case of material property jumps spatial fluxes remain continuous on vertical interfaces, so choosing spatial flux quantities in q will result in simpler solution and express of Riemann solutions.
2) Eventually we need spatial fluxes on vertical faces anyways (for most DGs we only care about the Riemann solution on vertical faces)





$$q = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \hline \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \vdots \end{bmatrix} \quad S$$

$$S \leftarrow \begin{pmatrix} \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{pmatrix}$$



$9_{9\times 1}$

$\dot{v} \leftarrow \rho - \nabla\cdot\tilde{\sigma} = \rho b$ 3 eqns $\times \frac{1}{\rho}$

$\dot{\sigma} \leftarrow \dot{\varepsilon} - \tilde{\nabla} v = 0$ 6 eqns $\times C^{-1}$ $\longrightarrow$

$$\dot{v} - \frac{1}{\rho}\nabla\cdot\sigma = b \quad (a)$$

$$\dot{\sigma} - C_{3\times3\times3\times3}\tilde{\nabla} v = 0 \quad (b)$$

(a) $\Rightarrow$

$$\dot{v}_1 - \frac{1}{\rho}\left(\sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3}\right) = b_1$$

$$\dot{v}_2 - \frac{1}{\rho}\left(\sigma_{21,1} + \sigma_{22,2} + \sigma_{23,3}\right) = b_2$$

$$\dot{v}_3 - \frac{1}{\rho}\left(\sigma_{31,1} + \sigma_{32,2} + \sigma_{33,3}\right) = b_3$$

S1, S4, S6 / S4, S2, S5 / S6, S5, S3

g_1 g_2 g_3

$$s = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix}$$

local coordinate system

$g_1 \to \frac{\partial}{\partial y_1}, \quad g_2 \to \frac{\partial}{\partial y_2}, \quad g_3 \to \frac{\partial}{\partial y_3}$

$$\begin{pmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{pmatrix} = C^y_{6\times6} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \end{bmatrix} \longrightarrow \begin{matrix} u_{1,1} \\ u_{2,2} \\ u_{3,3} \\ u_{1,2}+u_{2,1} \\ u_{2,3}+u_{3,2} \\ u_{3,1}+u_{1,3} \end{matrix} \longrightarrow \begin{pmatrix} b \\ \\ \end{pmatrix}$$

in y_1, y_2, y_3 system

$(C)^y_{3\times3\times3\times3} \longrightarrow (C)^x_{3\times3\times3\times3}$

$C^y_{ijkl} = Q_{im}Q_{jn}Q_{ko}Q_{lp}\, C^x_{mnop}$

want given

Voigt

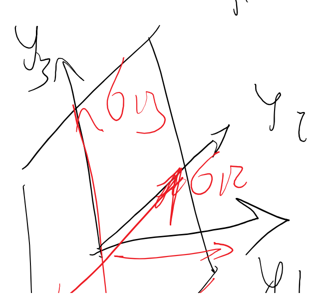
C in y system

$$\dot{S} - C \begin{bmatrix} v_{1,1} \\ v_{2,2} \\ v_{3,3} \\ v_{1,2}+v_{2,1} \\ v_{2,3}+v_{3,2} \\ v_{3,1}+v_{1,3} \end{bmatrix}$$

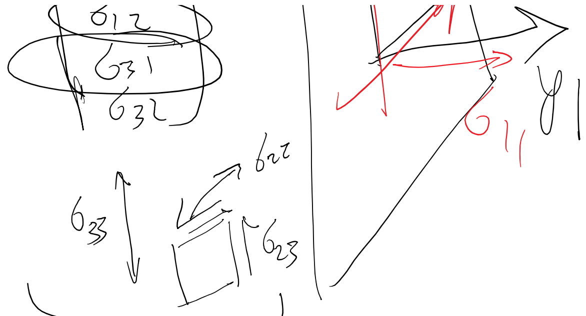
row 1

g_1 g_2 g_3

$S_1 \leftarrow \begin{matrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{31} \end{matrix}$ S4 S5



$(v_{3,1}+v_{1,3})$ s_4 s_5



do not appear in net spatial flux!

row 1

q_1 q_2 q_3

$$\dot{s}_1 = C_{11} v_{1,1} - C_{12} v_{2,2} - C_{13} v_{3,3} - C_{14}(v_{2,2}+v_{2,1}) - C_{15}(v_{2,3}+v_{3,2}) - C_{16}(v_{3,1}+v_{1,3}) = 0$$

row 4

$$\dot{s}_4 = C_{41} v_{1,1} - C_{42} v_{2,2} - C_{43} v_{3,3} - C_{44}(v_{2,2}+v_{2,1}) - C_{45}(v_{2,3}+v_{3,2}) - C_{46}(v_{3,1}+v_{1,3}) = 0$$

$$\dot{q} + A_1 q_{,1} + A_2 q_{,2} + A_3 q_{,3} = 0$$

$\underbrace{\partial q / \partial y_1}_{}$ red terms in □1, □2 — blue ones — green ones

we can form A_2, A_3 but eventually $q_{,2}$ & $q_{,3}$ are zero!

$$A_1 = \begin{array}{c|ccccccccc} & v_{1,1} & v_{2,1} & v_{3,1} & s_{1,1} & s_{2,1} & s_{3,1} & s_{4,1} & s_{5,1} & s_{6,1} \\ & & & & & \sigma_{22} & \sigma_{33} & & \sigma_{23} & \\ \hline v_1 & & & & -1/\rho & & & & & \\ v_2 & & & & & 0 & & -1/\rho & & \\ v_3 & & & & & & & & & -1/\rho \\ s_1 & C_{11} & C_{14} & C_{16} & & & & & & \\ s_2 & C_{21} & C_{24} & C_{26} & & & & & & \\ s_3 & C_{31} & C_{34} & C_{36} & & & & & & \\ s_4 & C_{41} & C_{44} & C_{66} & & & & & & \\ s_5 & C_{51} & C_{54} & C_{66} & & & & & & \\ s_6 & C_{61} & C_{64} & C_{66} & & & & & & \end{array}$$

$C_{i1}\ C_{i4}\ C_{i6}$

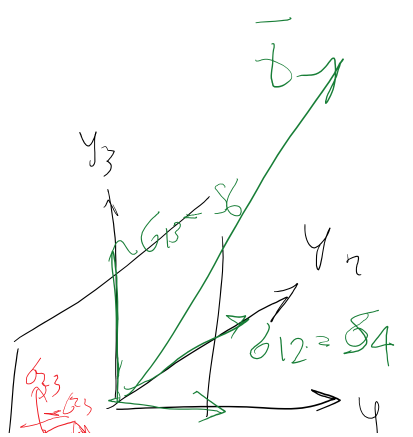
$$s = \begin{bmatrix} \sigma_{11} \to s_1 \\ \sigma_{22} \to s_2 \\ \sigma_{33} \to s_3 \\ \sigma_{12} \to s_4 \\ \sigma_{23} \to s_5 \\ \sigma_{31} \to s_6 \end{bmatrix}$$

9x9 A is given — find the Riemann sln

$v_1 \longleftrightarrow \sigma_{11}\ (s_1)$

are continuous

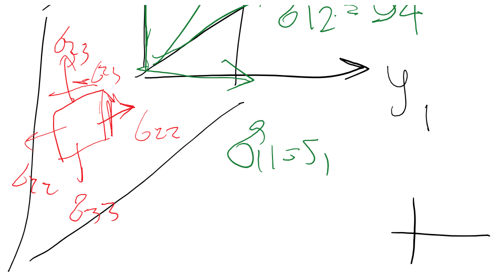
$v_2 \longleftrightarrow \sigma_{12}\ (s_4)$



are continuous $V_2 \leftrightarrow \sigma_{12}\ (S_4)$

$V_3 \leftrightarrow \sigma_{13}\ (S_6)$

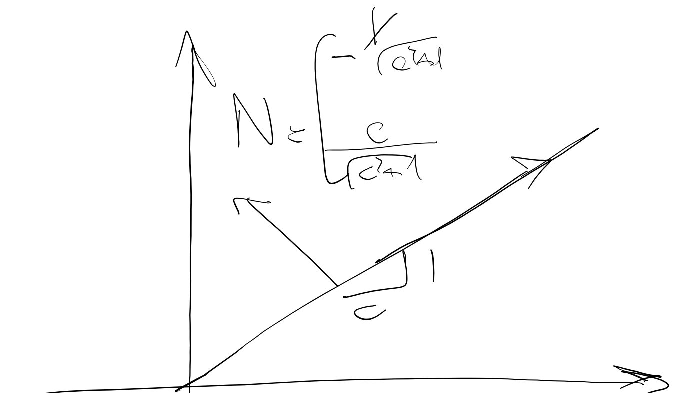
$\underbrace{\dot{E} - \tilde{\nabla} V = 0}_{\text{continuity of displacement}}$



$\sigma_{22}, \sigma_{33}, \sigma_{32}$

$\downarrow$ S_2, S_3, S_5

Can suffer jumps



$$N = \begin{bmatrix} -\frac{1}{\sqrt{c^2+1}} \\ \frac{c}{\sqrt{c^2+1}} \end{bmatrix}$$

$$[F] \cdot N = 0$$

f_{y}, f_t

$$[f_{y}]\, n_y + [f_t]\, n_t = 0$$

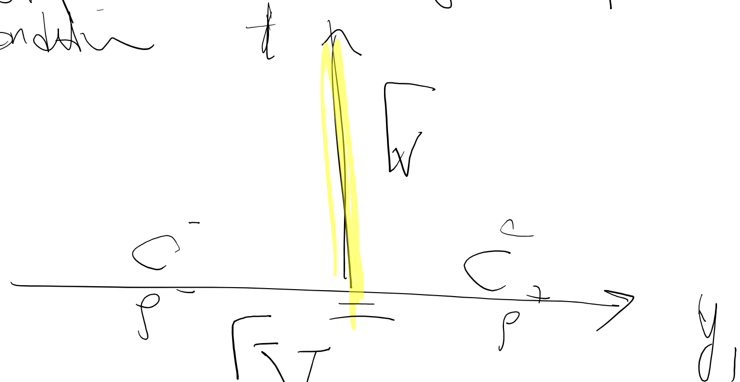
$$[f_{y}] \left(\frac{-1}{\sqrt{c^2+1}} \right) + [f_t] \frac{c}{\sqrt{c^2+1}} = 0$$

$$\boxed{[f_{y}] = c\,[f_t]}$$

$$\underbrace{\dot{q}}_{f_t = q} + \underbrace{A_1 q_{,y_1}}_{f_y = A_1 q} = 0$$

$$\boxed{[A q] = c\,[q]}$$

Jump condition



if $c = 0$

$$\boxed{[A q] = 0}$$

if A is conti

$$A[q] = c[q]$$

$[q]$ on Γ is an eigenvector

$A[q] = 0[q]$

$[q]$ is an eigenvector for zero eigenvalue

Reading assignment:

Available online at www.sciencedirect.com
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Comput. Methods Appl. Mech. Engrg. 329 (2018) 24–39

Computer methods in applied mechanics and engineering

www.elsevier.com/locate/cma

An exact Riemann solver for wave propagation in arbitrary anisotropic elastic media with fluid coupling

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$$[-\tilde{C}_{ij}] := \left[-\frac{\partial \tilde{h}_i}{\partial \tilde{q}_j}\right] = \begin{pmatrix} \mathbb{O}_{3\times3} & \mathbb{A}_{3\times6} \\ \mathbb{B}_{6\times3} & \mathbb{O}_{6\times6} \end{pmatrix}$$

Same matrix above

$$\mathbb{A} = -\begin{pmatrix} \tilde{D}_{15} & \tilde{D}_{25} & \tilde{D}_{35} & \tilde{D}_{45} & \tilde{D}_{55} & \tilde{D}_{56} \\ \tilde{D}_{14} & \tilde{D}_{24} & \tilde{D}_{34} & \tilde{D}_{44} & \tilde{D}_{45} & \tilde{D}_{46} \\ \tilde{D}_{13} & \tilde{D}_{23} & \tilde{D}_{33} & \tilde{D}_{34} & \tilde{D}_{35} & \tilde{D}_{36} \end{pmatrix} \tag{15}$$

and

$$\mathbb{B} = -\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/\rho \\ 0 & 1/\rho & 0 \\ 1/\rho & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{16}$$

With the consideration of heterogeneous material on the both sides of the interface, (14) has 9 eigenvalues, which are comprised of 3 negative , 3 zero and 3 positive eigenvalues as

$$\Lambda = diag(-\mathbb{E}_1, -\mathbb{E}_2, -\mathbb{E}_3, 0, 0, 0, +\mathbb{E}_1^+, +\mathbb{E}_2^+, +\mathbb{E}_3^+) \tag{17}$$

where the superscript "+" means a variable from the opposite side of the interface. Since $-\tilde{C}$ is a block anti-diagonal matrix, the 3 non-zero eigenvalue square matrix $\mathbb{E}^2_{3\times3}$ can be solved from

$$\mathbb{M}_{3\times3} = \mathbb{A}_{3\times6}\mathbb{B}_{6\times3} = \begin{pmatrix} \frac{\tilde{D}_{55}}{\rho} & \frac{\tilde{D}_{45}}{\rho} & \frac{\tilde{D}_{35}}{\rho} \\ \frac{\tilde{D}_{45}}{\rho} & \frac{\tilde{D}_{44}}{\rho} & \frac{\tilde{D}_{34}}{\rho} \\ \frac{\tilde{D}_{35}}{\rho} & \frac{\tilde{D}_{34}}{\rho} & \frac{\tilde{D}_{33}}{\rho} \end{pmatrix} \tag{18}$$

whose eigenvector matrix is $\mathbb{R}_{3\times3}$. We have also successfully applied this good property to the large system matrix in viscoelastic wave modeling [41]. We can conveniently perform diagonalization:

$$-\tilde{C} = \frac{1}{2}\begin{pmatrix} \mathbb{R} & \mathbb{R} \\ -\mathbb{B}\mathbb{R}\mathbb{E}^{-1} & \mathbb{B}\mathbb{R}\mathbb{E}^{-1} \end{pmatrix}\begin{pmatrix} -\mathbb{E} & \\ & \mathbb{E} \end{pmatrix}\begin{pmatrix} \mathbb{R}^{-1} & -\mathbb{E}^{-1}\mathbb{R}^{-1}\mathbb{A} \\ \mathbb{R}^{-1} & \mathbb{E}^{-1}\mathbb{R}^{-1}\mathbb{A} \end{pmatrix}. \tag{19}$$

2006_Reza_Abedi_SDG_Elastodynamics.pdf

Appendix B. Definitions of Godunov values

The following subsections present expressions for the Godunov values of stress and velocity. The Godunov strains are obtained by applying the inverse constitutive relation to the Godunov stresses, and the Godunov momentum densities are obtained by applying the forward constitutive relation to the Godunov velocities.

B.1. Godunov values for solutions on $\mathbb{E}^1 \times \mathbb{R}$

Let $c = \sqrt{E/\rho}$ denote the elastic wave speed in which E denotes Young's modulus. We drop all subscripts in this section since there is only one spatial direction. Consider a causal patch in $\mathbb{E}^1 \times \mathbb{R}$, as shown in Fig. B.1(a). The Godunov values of the mechanical fields on the noncausal interface $\Gamma_{\alpha\beta}$ are functions of the fields on the adjacent elements, Q_α and Q_β:

$$\sigma^G = \frac{1}{2}(\sigma^\alpha + \sigma^\beta) + \frac{E}{2c}(\dot{u}^\beta - \dot{u}^\alpha), \tag{B.1a}$$

$$\dot{u}^G = \frac{1}{2}(\dot{u}^\alpha + \dot{u}^\beta) + \frac{c}{2E}(\sigma^\beta - \sigma^\alpha). \tag{B.1b}$$

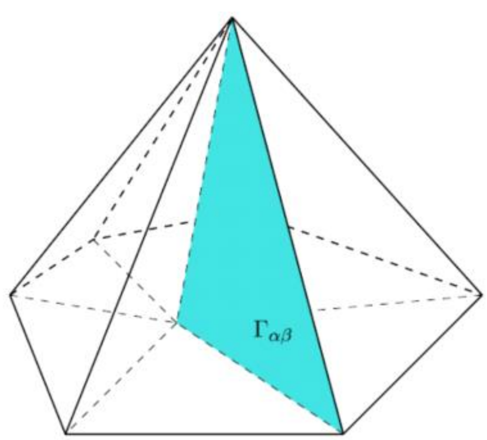

Fig. B.2. Causal patch in $\mathbb{E}^2 \times \mathbb{R}$ with noncausal interface $\Gamma_{\alpha\beta}$.

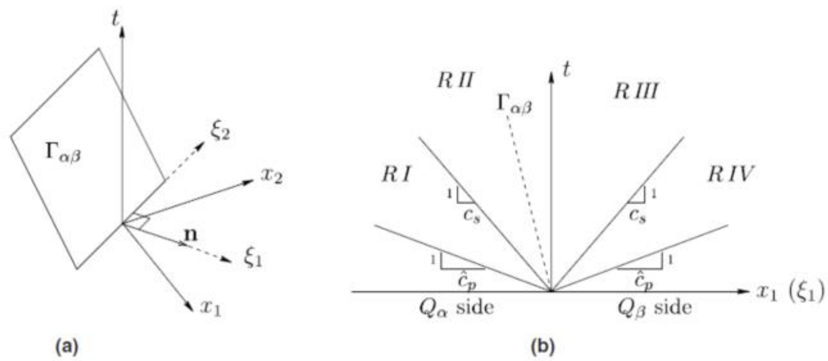

Fig. B.3. Coordinates and inclination of a noncausal interface in $\mathbb{E}^2 \times \mathbb{R}$: (a) local coordinates on noncausal interface $\Gamma_{\alpha\beta}$, (b) regions (*RI*–*RIV*) for classifying the inclination of the interface $\Gamma_{\alpha\beta}$.