DG2020/04/13

Monday, April 13, 2020 11:44 AM

$$
A_1 = \begin{array}{c|ccccccccc}
 & v_{1,1} & v_{2,1} & v_{3,1} & S_{1,1} & \overset{\sigma_{22}}{S_{2,1}} & \overset{\sigma_{33}}{S_{3,1}} & S_{4,1} & \overset{\sigma_{23}}{S_{5,1}} & S_{6,1} \\
\hline
v_1 & & & & -\frac{1}{\rho} & 0 & & & & \\
v_2 & & & & & & & -\frac{1}{\rho} & & \\
v_3 & & & & & & & & & -\frac{1}{\rho} \\
S_1 & C_{11} & C_{14} & C_{16} & & & & & & \\
S_2 & C_{21} & C_{24} & C_{26} & & & & & & \\
S_3 & C_{31} & C_{34} & C_{36} & & & & & & \\
S_4 & C_{41} & C_{44} & C_{66} & & & & & & \\
S_5 & C_{51} & C_{54} & C_{66} & & & & & & \\
S_6 & C_{61} & C_{64} & C_{66} & & & & & &
\end{array}
$$

For isotropic elasticity:

$$
C_{ijkl} = \underbrace{\lambda \delta_{ij}\delta_{kl} + \mu}_{\text{Lamé's parameters}} (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})
$$

$$
\text{Voigt stiffness } C_{6\times 6} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & & & \\ & \lambda+2\mu & \lambda & & 0_{3\times 3} & \\ & & \lambda+2\mu & & & \\ & & & \mu & 0 & 0 \\ & & & & \mu & 0 \\ & & & & & \mu \end{bmatrix}
$$

For isotropic elasticity, we have the spatial flux matrix as,

$$
A_1 = \begin{array}{c|ccccccccc}
 & v_{1,1} & v_{2,1} & v_{3,1} & S_{1,1} & S_{2,1} & S_{3,1} & S_{4,1} & S_{5,1} & S_{6,1} \\
\hline
v_1 & & & & -\frac{1}{\rho} & & & & & \\
v_2 & & & & & & & -\frac{1}{\rho} & & \\
v_3 & & & & & & & & & -\frac{1}{\rho} \\
S_1 & \lambda+2\mu & & & & & & & & \\
S_2 & \lambda & & & & & & & & \\
S_3 & \lambda & & & & & & & & \\
S_4 & & \mu & & & & & & & \\
S_5 & & & & & & & & & \\
S_6 & & & \mu & & & & & &
\end{array}
$$

$v_1 \longleftrightarrow S_1 \quad \sigma_{11} \longrightarrow$ longitudinal/dilatational mode $c_d = \sqrt{\frac{\lambda+2\mu}{\rho}}$

$v_2 \longleftrightarrow S_4 \quad \sigma_{12} \Big\}$ shear mode $c_s = \sqrt{\frac{\mu}{\rho}}$

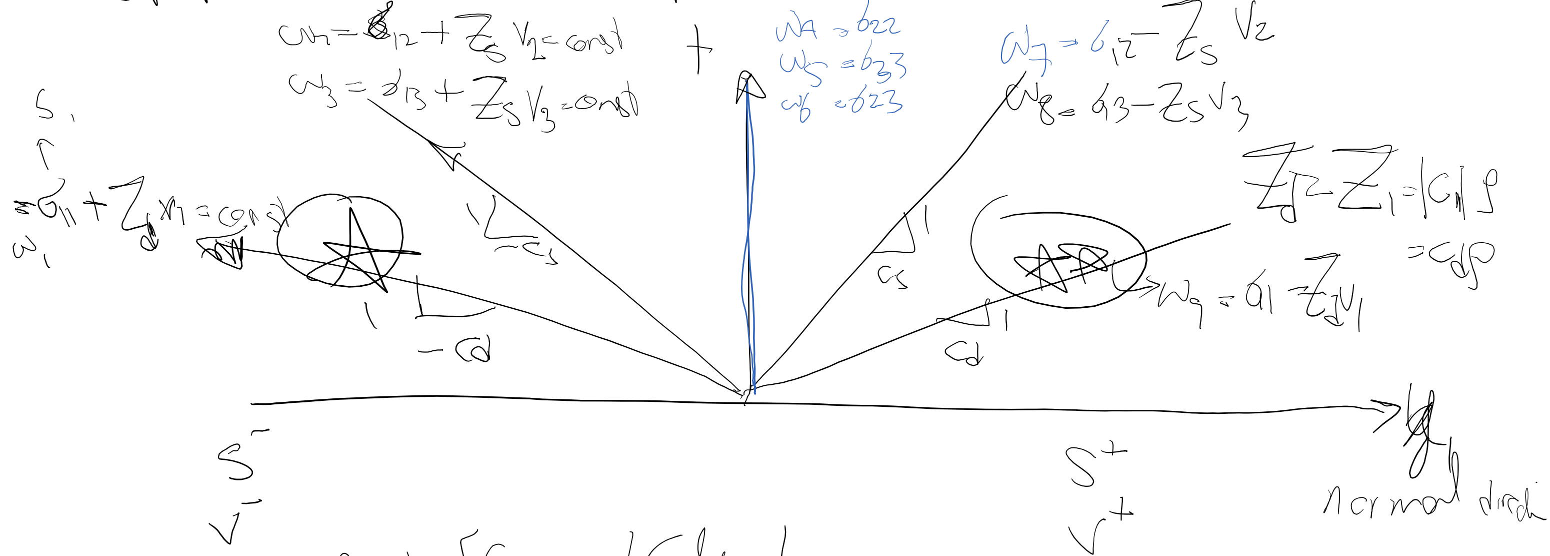
$v_2 \longleftrightarrow S_4 \quad \sigma_{12}$

$v_3 \longleftrightarrow S_6 \quad \sigma_{13}$ shear modes $c_s = \sqrt{\frac{\mu}{\rho}}$

$c_d > c_s$

$A_{9\times9} \longrightarrow$ 9 eigenvalues

the eigenvalues of A which are the wave speeds $\{-c_d, -c_s, -c_s, 0, 0, 0, c_s, c_s, c_d\}$



$$\underbrace{\begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_9 \end{bmatrix}}_{L} A = \begin{bmatrix} c_1 l_1 \\ c_2 l_2 \\ \vdots \\ c_n l_n \end{bmatrix} = \begin{bmatrix} c_1 & & \\ & \ddots & \\ & & c_9 \end{bmatrix} \begin{bmatrix} l_1 \\ \vdots \\ l_9 \end{bmatrix} \longleftarrow l_i A = c_i l_i$$

row vector

Left eigenvalue problem

$$\underbrace{LA = CL}$$

$A^t l_i^t = c_i l_i^T$

column vector

l eigenvector = right eigenvector of A^t

left & right eigenvalues are the same (here they are the wave speeds)

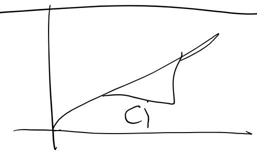
$L \times \dot q + A q_{,y_1} = 0$

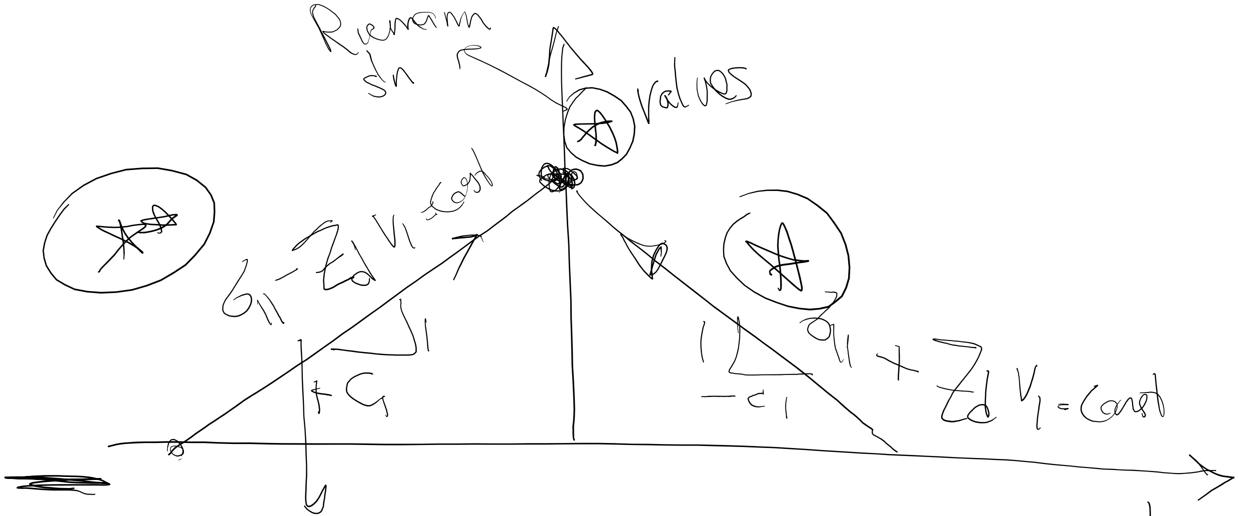
$(\dot{Lq}) + \underbrace{LA}_{CL} q_{,y_1} = 0 \qquad \underbrace{(\dot{Lq})} + C(Lq)_{,y_1} = 0$

$$\boxed{w = Lq}$$

$\dot w + C w_{,y_1} = 0 \longrightarrow \boxed{\dot w_i + c_i w_{i,y_1} = 0}$ no sum on i

$\omega + c\,\omega_{,y_1} = 0 \to \omega_i + c_i\,\omega_{i,y_1} = 0$ no sum on i

ω_i is constant on  lines



Z_d^+ -1

Z_d^- $+1$

$$\sigma_{11}^* - Z_d^- v_1^* = \sigma_{11}^- - Z_d^- v_1^-$$

$$\sigma_{11}^* + Z_d^+ v_1^* = \sigma_{11}^+ + Z_d^+ v_1^+$$

add them

$$v_1^* = \frac{(\sigma_{11}^+ - \sigma_{11}^-) + Z_d^+ v_1^+ + Z_d^- v_1^-}{Z_d^+ + Z_d^-}$$

$$\sigma_{11}^* = \frac{(\sigma_{11}^- Z_d^+ + \sigma_{11}^+ Z_d^-) + Z_d^- Z_d^+ (v_1^+ - v_1^-)}{Z_d^- + Z_d^+}$$

in fact

$$v_i^* = \frac{(\sigma_{1i}^+ - \sigma_{1i}^-) + Z_i^+ v_i^+ + Z_i^- v_i^-}{Z_i^+ + Z_i^-}$$

$Z_1 = c_d\rho$

$Z_2 = Z_3 = c_s\rho$

$$\sigma_{1i}^* = \frac{(\sigma_{1i}^+ Z_i^- + \sigma_{1i}^- Z_i^+) + Z_i^- Z_i^+ (v_i^+ - v_i^-)}{Z_i^- + Z_i^+}$$

General 3D/2D

different material

general 3D/2D
s'n different material $Z_i^- + Z_i^+$

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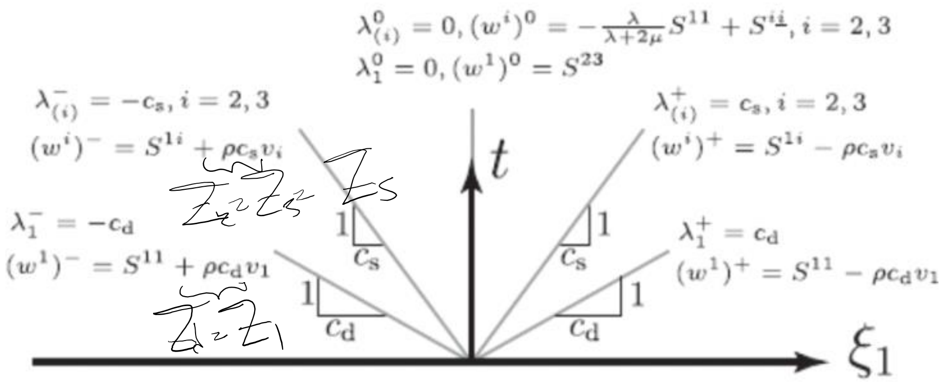
$$
\mathrm{q} := \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ S^{11} \\ S^{12} \\ S^{13} \\ S^{22} \\ S^{23} \\ S^{33} \end{bmatrix}, \quad
\mathrm{R} := \begin{bmatrix} \rho b^1 \\ \rho b^2 \\ \rho b^3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad
\mathrm{A}_1 := -\begin{bmatrix}
0 & 0 & 0 & 1/\rho & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1/\rho & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1/\rho & 0 & 0 & 0 \\
\lambda+2\mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mu & 0 & 0 & 0 & 0 & 0 & 0 \\
\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \tag{4}
$$



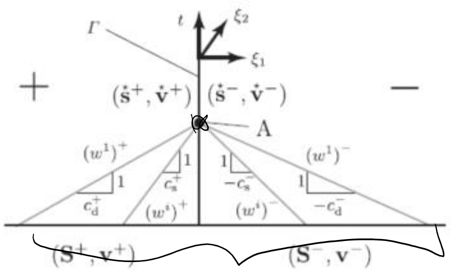
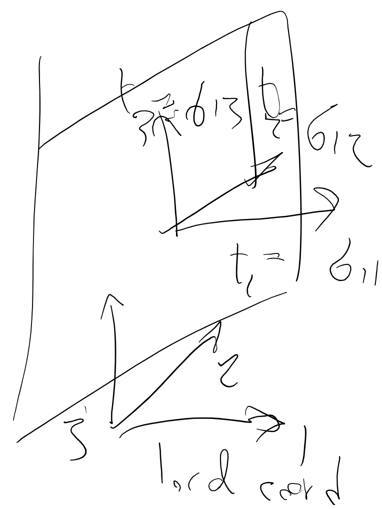


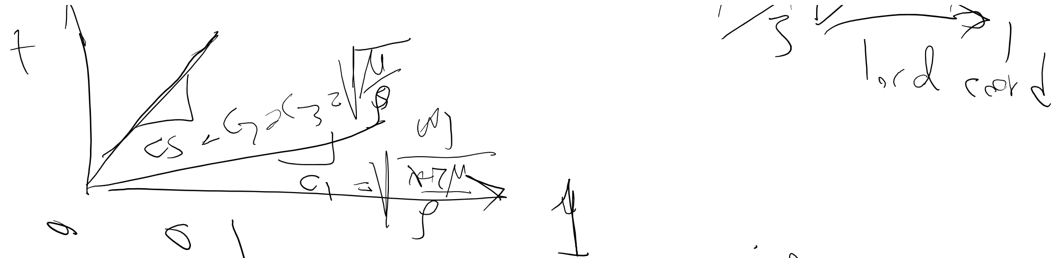
Fig. 3. Riemann problem, projected into the ξ_1–t plane, for distinct initial data on opposing sides, + and −, of the trajectory Γ of a generic material interface in 2d × time. By convention, the ξ_1-direction in the local coordinate frame (ξ_i, t) aligns with the outward spatial normal vector for the region on the + side of Γ; *cf.* Fig. 1.

for isotropic material

$$\underset{\text{traction}}{w_{3\times1}} = t_{3\times1} - \underbrace{Z_{3\times3}}_{\text{impedence matrice}} \underset{\text{velocity}}{V_{3\times1}}$$

t $\sqrt{\mu/\rho}$





$$Z = \begin{bmatrix} \rho c_d & 0 & 0 \\ 0 & \rho c_s & 0 \\ 0 & 0 & \rho c_s \end{bmatrix}$$ for isotropic material

$t_i - \rho c_i V_i$ constant on c_i lines

What about anisotropic

Z not diagonal so $t_1 = \sigma_{11}$ is coupled to V_2 & V_3 as well

$$t_i - \sum_j Z_{ij} V_j = \text{constant}$$

Riemann solutions and spacetime discontinuous Galerkin method for linear elastodynamic contact ☆

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^ it's a paper for contact/friction (using Riemann solutions) for isotropic material.

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An exact Riemann solver for wave propagation in arbitrary anisotropic elastic media with fluid coupling

Qiwei Zhan[a,b], Qiang Ren[c], Mingwei Zhuang[a,d], Qingtao Sun[a], Qing Huo Liu[a,*]

Bonded solutions (e.g. no friction or contact) but for anisotropic solid.

...
The full 3x3 impedance matrix:

where the superscript "+" means a variable from the opposite side of the interface. Since $-\bar{C}$ is a block anti-diagonal matrix, the 3 non-zero eigenvalue square matrix $\mathbb{E}^2_{3\times3}$ can be solved from

$$\mathbb{M}_{3\times3} = \mathbb{A}_{3\times6}\mathbb{B}_{6\times3} = \begin{pmatrix} \frac{\tilde{D}_{55}}{\rho} & \frac{\tilde{D}_{45}}{\rho} & \frac{\tilde{D}_{35}}{\rho} \\ \frac{\tilde{D}_{45}}{\rho} & \frac{\tilde{D}_{44}}{\rho} & \frac{\tilde{D}_{34}}{\rho} \\ \frac{\tilde{D}_{35}}{\rho} & \frac{\tilde{D}_{34}}{\rho} & \frac{\tilde{D}_{33}}{\rho} \end{pmatrix} \quad (18)$$

where the superscript "+" means a variable from the opposite side of the interface. Since $-\tilde{C}$ is a block anti-diagonal matrix, the 3 non-zero eigenvalue square matrix $\mathbb{E}^2_{3\times3}$ can be solved from

$$\mathbb{M}_{3\times3} = \mathbb{A}_{3\times6}\mathbb{B}_{6\times3} = \begin{pmatrix} \frac{\tilde{D}_{55}}{\rho} & \frac{\tilde{D}_{45}}{\rho} & \frac{\tilde{D}_{35}}{\rho} \\ \frac{\tilde{D}_{45}}{\rho} & \frac{\tilde{D}_{44}}{\rho} & \frac{\tilde{D}_{34}}{\rho} \\ \frac{\tilde{D}_{35}}{\rho} & \frac{\tilde{D}_{34}}{\rho} & \frac{\tilde{D}_{33}}{\rho} \end{pmatrix} \tag{18}$$

Another way or solving the Riemann solutions is through the jump conditions (rather than using characteristics)

last time I shared

$$q_t + A q_{x,y} = 0$$

$$A[v_1 | v_2 | \cdots | v_n] = [v_1 | v_2 | \cdots v_n] \begin{bmatrix} c_1 & & \\ & \ddots & \\ & & c_n \end{bmatrix}$$

right eigenvectors

eigenvalues

$A v_i = c_i v_i$

Region $i-1$, i

t, c_i, y

$q^{(i)} - q^{(i-1)} = \alpha^i v^i$

unknown

add these together

$q^+ - q^0 = \alpha^1 v^1$

$q^+ - q^i = \alpha^i v^i$

$q^1 = q^0 + \alpha^1 v^1$

q^i, q^+, $n+1$

$$q^T - q^- = \sum_{i=1}^{n} \alpha^i v^i$$

$$= [v_1 | \cdots | v_n] \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$$

$$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix} = V^{-1}(q^+ - q^-)$$

once α's are known $\rightarrow$

we find the s'n in all sectors

$-c_s, c_s$

RI

$v_1 \& S_1(\sigma_{11})$ wall

RII

$v_2 \& \sigma_{12}(S_4)$

$v_3 \& \sigma_{13}(S_6)$

t

$\sigma_{22} = S_2$

$\sigma_{33} = S_3$

$\sigma_{23} = S_5$

jump

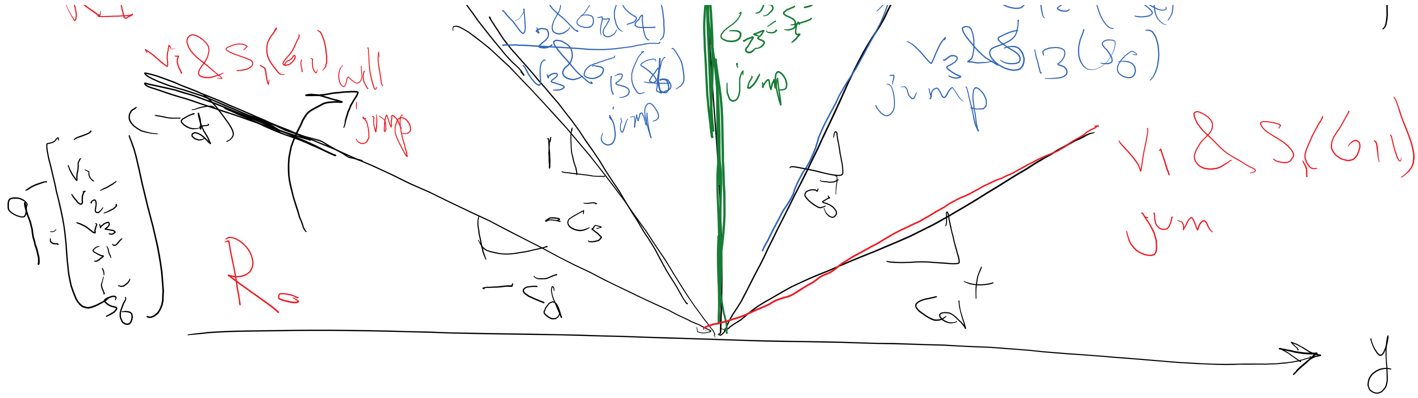
RIII

RIV

$v_2 \& \sigma_{12}(S_4)$

$v_3 \& \sigma_{13}(S_6)$

jump



A space–time discontinuous Galerkin method for linearized elastodynamics with element-wise momentum balance

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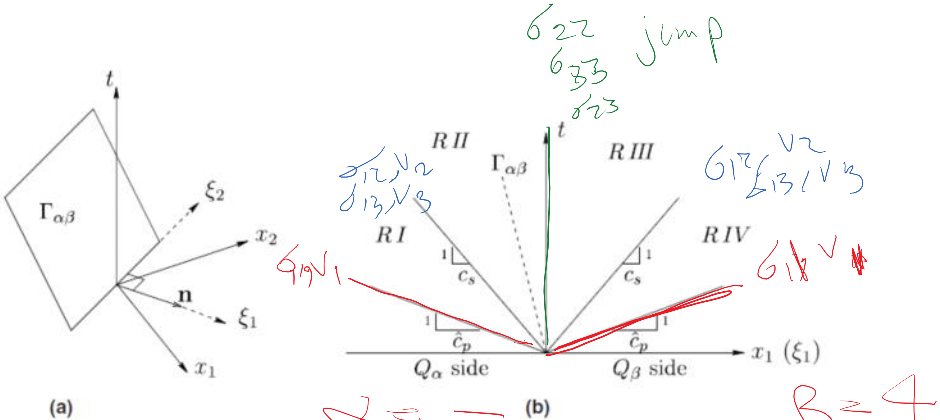


Fig. B.3. Coordinates and inclination of a noncausal interface in $\mathbb{E}^2 \times \mathbb{R}$: (a) local coordinates on noncausal interface $\Gamma_{\alpha\beta}$, (b) regions (RI–RIV) for classifying the inclination of the interface $\Gamma_{\alpha\beta}$.

$$(\sigma^{11})^G = \frac{1}{2}\{(\sigma^{11})^\alpha + (\sigma^{11})^\beta\} + \frac{\rho\hat{c}_p}{2}[\dot{u}_1] \qquad \text{All regions,} \tag{B.3a}$$

$$(\sigma^{22})^\alpha = \begin{cases} (\sigma^{22})^\alpha + \dfrac{\hat{\lambda}}{2(\hat{\lambda}+2\mu)}[\sigma^{11}] + \dfrac{\hat{\lambda}}{2\hat{c}_p}[\dot{u}_1] & \text{Regions I and II,} \\ (\sigma^{22})^\beta - \dfrac{\hat{\lambda}}{2(\hat{\lambda}+2\mu)}[\sigma^{11}] + \dfrac{\hat{\lambda}}{2\hat{c}_p}[\dot{u}_1] & \text{Regions III and IV,} \end{cases} \tag{B.3b}$$

Same material on both sides (2D)

$$(\sigma^{12})^G = \begin{cases} (\sigma^{12})^\alpha & \text{Region I,} \\ \dfrac{1}{2}\{(\sigma^{12})^\alpha + (\sigma^{12})^\beta\} + \dfrac{\rho c_s}{2}[\dot{u}_2] & \text{Regions II and III,} \\ (\sigma^{12})^\beta & \text{Region IV,} \end{cases} \tag{B.3c}$$

V_1^*

$$\dot{u}_1^G = \frac{1}{2}(\dot{u}_1^\alpha + \dot{u}_1^\beta) + \frac{1}{2\rho\hat{c}_p}[\sigma^{11}] \qquad \text{All regions,} \tag{B.3d}$$

$$\dot{u}_2^G = \begin{cases} \dot{u}_2^\alpha & \text{Region I,} \\ \dfrac{1}{2}(\dot{u}_2^\alpha + \dot{u}_2^\beta) + \dfrac{1}{2\rho c_s}[\sigma^{12}] & \text{Regions II and III,} \\ \dot{u}_2^\beta & \text{Region IV,} \end{cases} \tag{B.3e}$$

in which

$$[f] = f^\beta - f^\alpha \tag{B.4}$$

$F^* \cdot N =$

$f_x^* \cdot n_x + f_t^* n_t$

$= \boxed{f_x^* n_x} + 0$ on vertical lines

$f_x^* = \sigma^*_{3\times3}$

$f_x^* n_x = \sigma^*_{3\times3} n_x = t^*_{3\times1} = \begin{bmatrix} \sigma_{11}^* \\ \sigma_{12}^* \\ \sigma_{13}^* \end{bmatrix}$

V_1^*

$$\dot u_1^G = \frac{1}{2}(\dot u_1^\alpha + \dot u_1^\beta) + \frac{1}{2\rho \hat c_p}[\sigma^{11}] \quad \text{All regions,} \tag{B.3d}$$

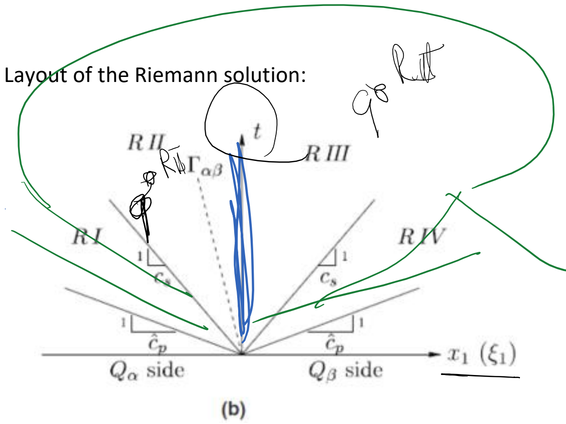
$$\dot u_2^G = \begin{cases} \dot u_2^\alpha & \text{Region I,} \\ \frac{1}{2}(\dot u_2^\alpha + \dot u_2^\beta) + \frac{1}{2\rho c_s}[\sigma^{12}] & \text{Regions II and III,} \\ \dot u_2^\beta & \text{Region IV,} \end{cases} \tag{B.3e}$$

in which

$$[f] = f^\beta - f^\alpha. \tag{B.4}$$

$t^*_{3\times1} = \sigma_{3\times3}\, n_x = \begin{bmatrix} \sigma_{11}^* \\ \sigma_{12}^* \\ \sigma_{13}^* \end{bmatrix}$

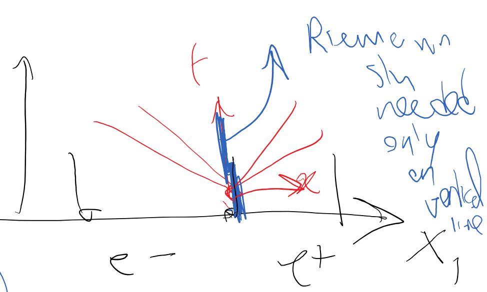
Layout of the Riemann solution:



(b)

is not needed for conventional time marching DG

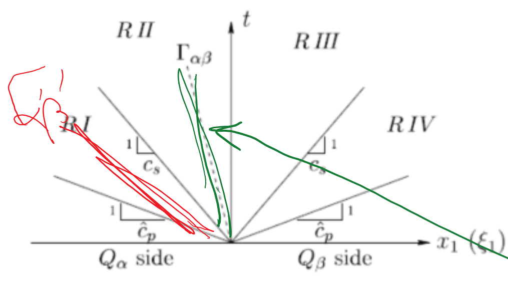
$f(q^*) = A q^*$ Unique on t axis

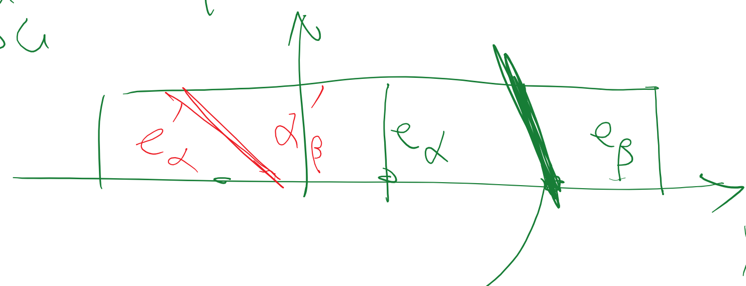


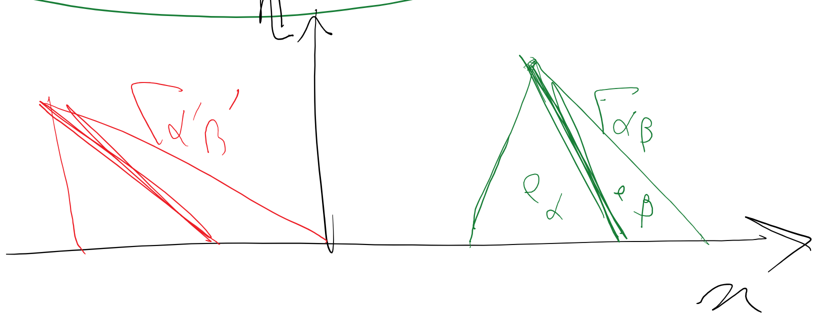
need RII, RIII s/n only

for conventional DG Riemann s/n only needed on vertical line

Spacetime DG



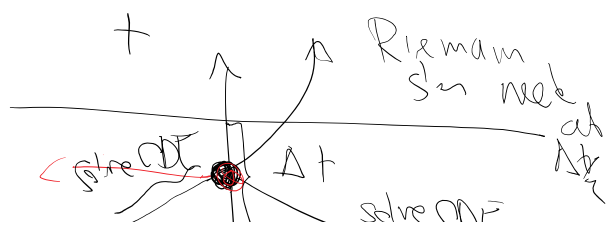




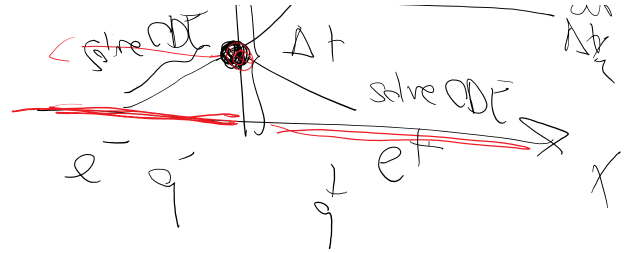
With spacetime DG methods we need to have the capability to have the Riemann solution in all Regions as opposed to only vertical line that is needed for space DG + time marching.

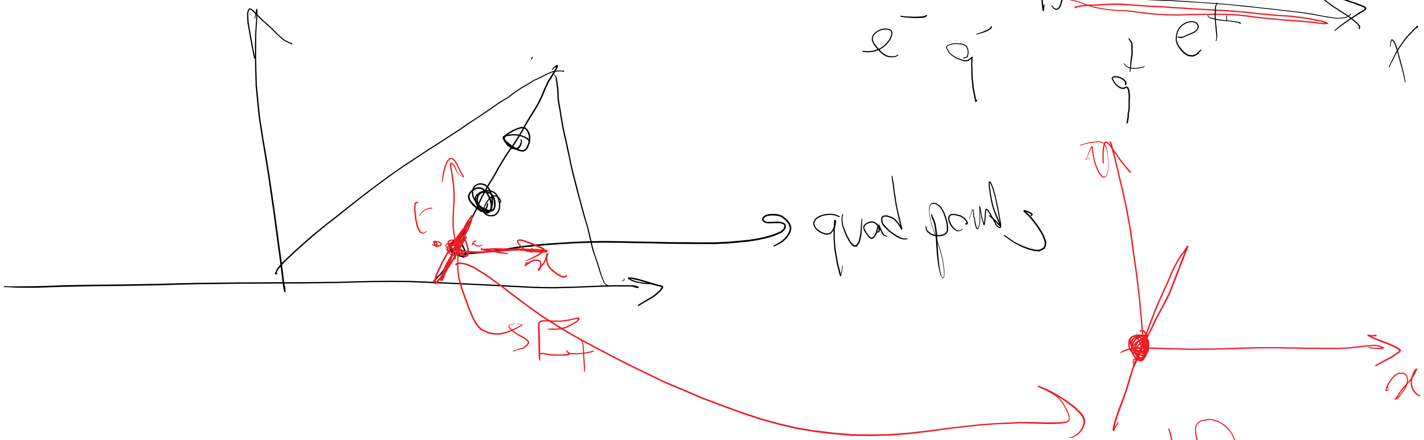
Final point, if we have source term, the solution is not constant along characteristics, and we need to solve ODEs along characteristics:(

spacetime DGs don't have this problem



don't have ...





Riemann sln @ origin → even with source term no ODE needs to be solved (ignore source term)

Next time:
Approximate Riemann solvers for nonlinear conservation laws (e.g. Burger's equation, Euler's equations, …)

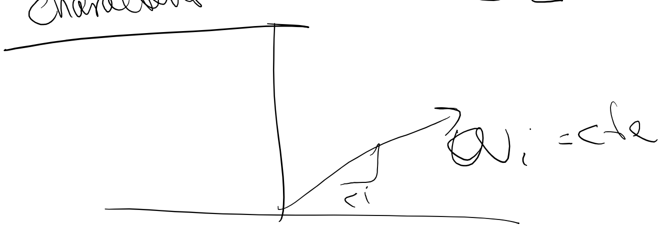
1D

$$\underbrace{U_t}_{q_t} + \underbrace{f(U)_{,x}}_{q_x} = 0$$

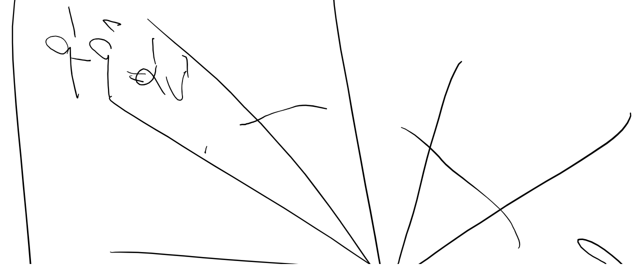
if linear $\quad f(U) = AU \quad\quad U_t + AU_{,x} = 0$

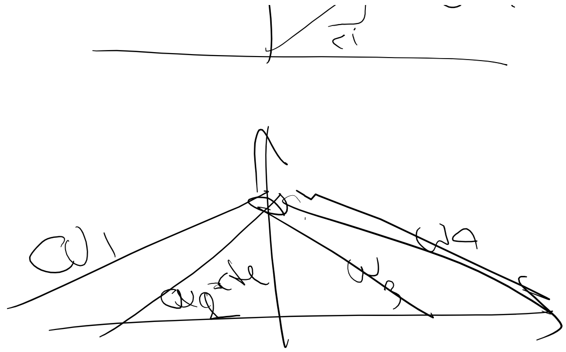
we know how to solve this

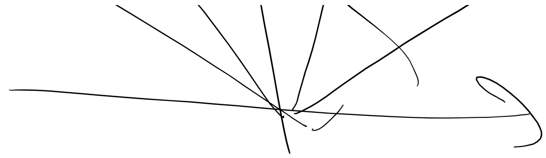
characteristics $\quad LA = \Lambda L$



jump approach

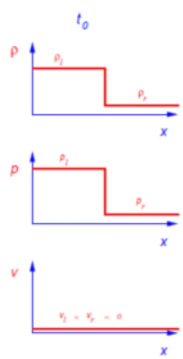


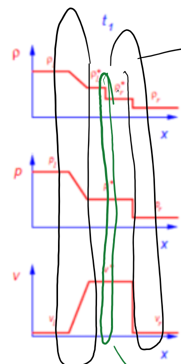




Cannot use either one for nonlinear PDE

Example of nonlinear 1D hyperbolic PDE (1D Euler's equation)





later time

shock

rarefac.

contact discontinuity (like linear PDE jump)

rarefaction wave