DG2020/04/15

Wednesday, April 15, 2020 11:41 AM

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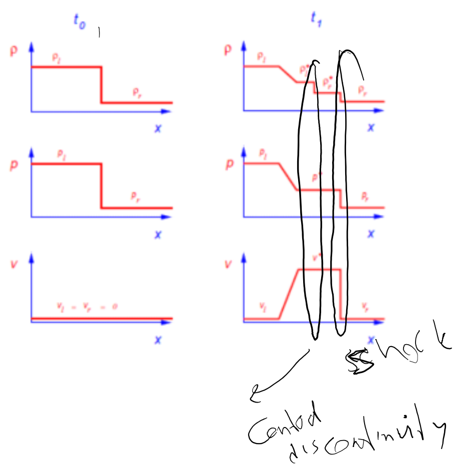
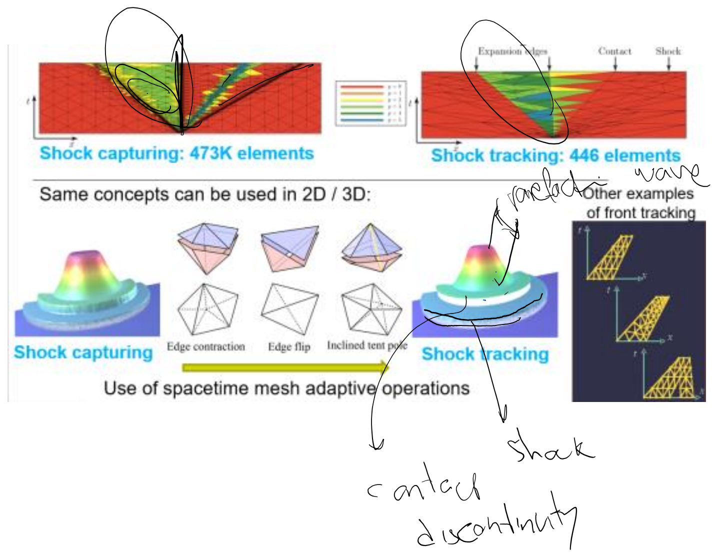
cws06_steiner_riemann.pdf



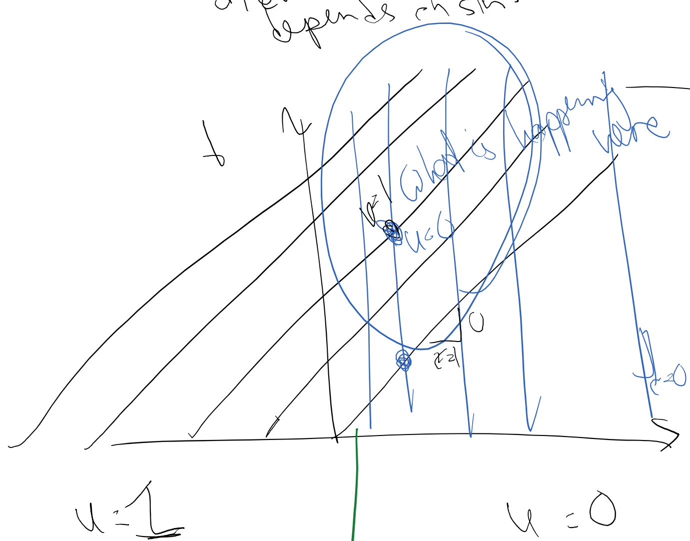


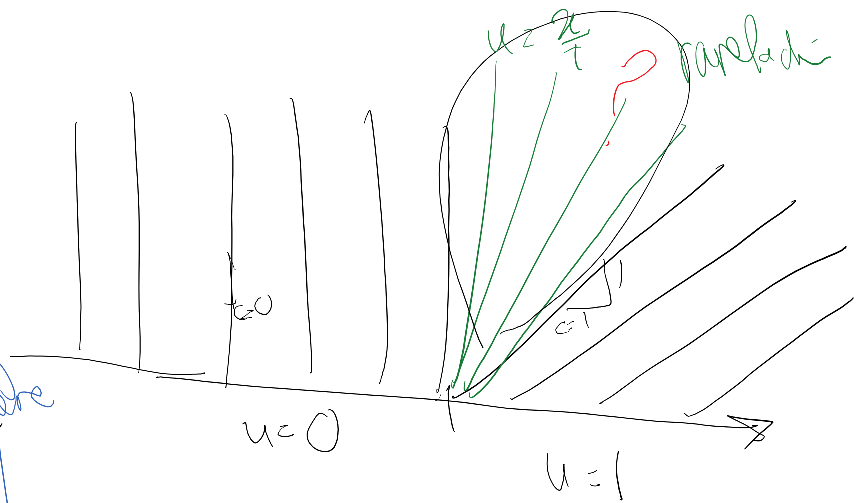
Burgers eqn — f_x is nonlinear in u

$$\dot{u} + \left(\frac{1}{2}u^2\right)_{,x} = 0$$

$$\dot{u} + u\,u_{,x} = 0$$

speed of characteristic depends on sln.



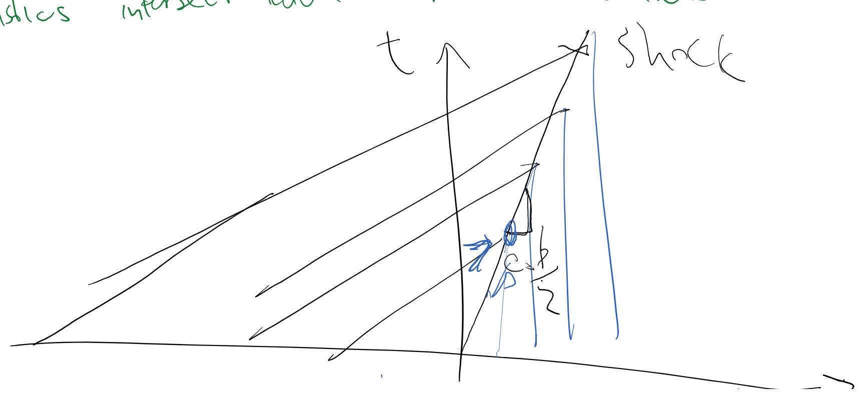


IC

$u_L < u_R$

$A(u) = u > 0$

Characteristics intersect each other → Shock

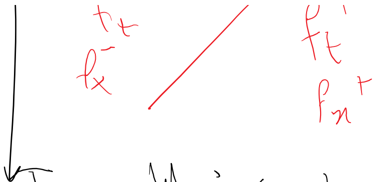


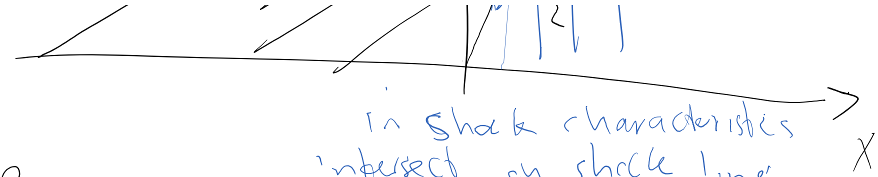
Balance law

$$f_t + f_{x,x} = 0$$

$\bar{f}_t$ f_t^+

f_x^- f_x^+

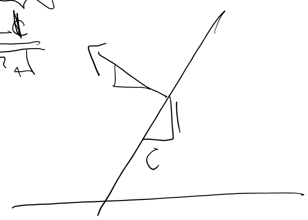




in shock characteristics intersect on shock line

Jump condition: $n_t[f_t] + n_x[f_x] = 0$

$n_t = \frac{c}{\sqrt{c^2+1}}$, $\quad n_x = \frac{-1}{\sqrt{c^2+1}}$



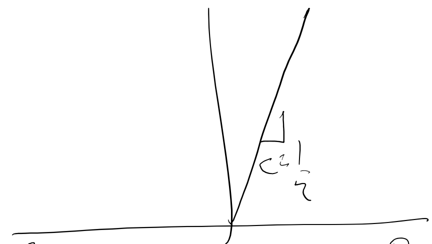
$$\boxed{[f_x] = c[f_t]}$$

Rankine Hugoniot (jump) condition

$$[f_x] = c[f_t]$$

$$\left(0 - \frac{1}{2}\right) = c(0 - 1)$$

$$\rightarrow \boxed{c = \frac{1}{2}}$$

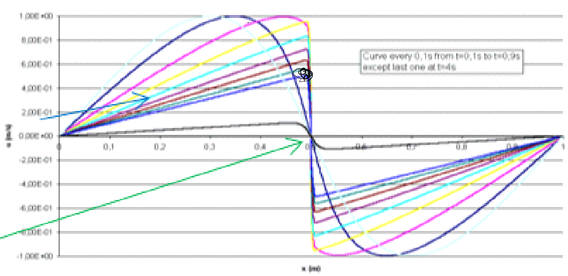


$f_t^- = u = 1$
$f_x^- = \frac{1}{2}u^2 = \frac{1}{2}$

$f_t^+ = u = 0$
$f_x^+ = \frac{1}{2}u^2 = 0$

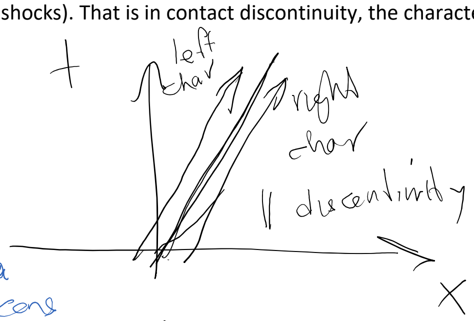
t = 0, smooth solution

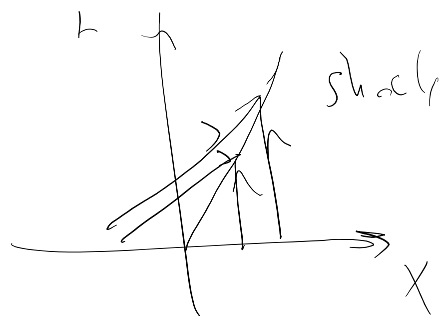
t > 0, shock has formed



Burger's equation $\quad u_t + \left(\frac{1}{2}u^2\right)_x = 0$

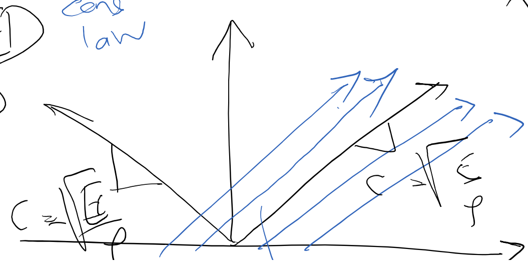
Difference between shocks and contact discontinuities in general: In contact discontinuity, the characteristics across discontinuity are parallel (they don't collide as in shocks). That is in contact discontinuity, the characteristics are like how we have them for LINEAR conservation laws across discontinuity.

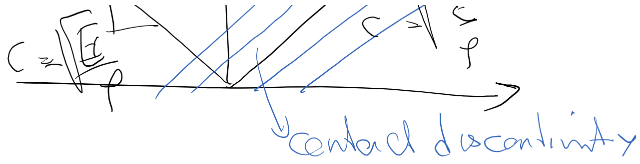




linear cons law

1D





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$$U_t + f(U)_x = 0. \tag{4.1}$$

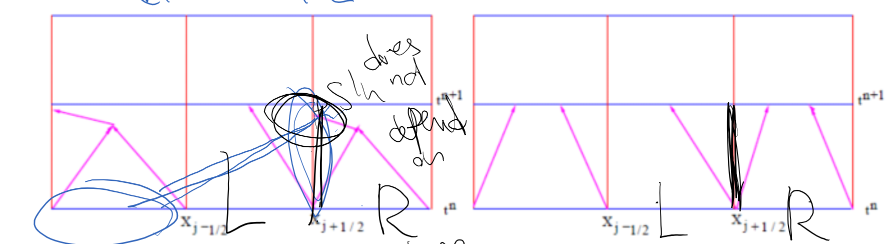
The discussion on the linear transport equation

$$U_t + aU_x = 0 \tag{4.2}$$

Riemann sln for 4.1

violate CFL

Burgers eqn $f(U) = \frac{U^2}{2}$



(a) Interacting Riemann waves. (b) Non-interacting waves.

FIGURE 4.3. Left: Waves of Riemann problems from neighboring interface can interact after some time. Right: The waves can be prevented from interacting before time Δt by the CFL condition (4.9)

Arnold_2000_Discontinuous Galerkin methods for elliptic problems.pdf

1. Locality. Let $K = K_1$ be an element in the triangulation, and let e be one of its edges. Assume first that e is an interior edge of our triangulation, so that there is a second element K_2 sharing the edge e with K_1. We then assume that $h_\sigma^{e,K}$ and $h_u^{e,K}$ depend on the restrictions $u_h|_{K_i}$ and $\sigma_h|_{K_i}$ of u_h and σ_h to K_i, $i = 1, 2$. More precisely, locality means that

$$h_\sigma^{e,K} = h_\sigma^{e,K}(\underbrace{u_h|_{K_1}, \sigma_h|_{K_1}}_{\text{left}}, \underbrace{u_h|_{K_2}, \sigma_h|_{K_2}}_{\text{right}}).$$

Actually, in all our examples, this fucntional dependence will have a special form in that both $h_\sigma^{e,K}$ and $h_u^{e,K}$ will depend only on the traces of $u_h|_{K_i}$, $\nabla u_h|_{K_i}$, and $\sigma_h|_{K_i}$ on the edge e. Since u_h, ∇u_h, and σ_h will, in general, be discontinuous across e, the trace of $u_h|_{K_1}$ on e will be different from the trace of $u_h|_{K_2}$ on e, and similarly ∇u_h and σ_h will each have two different traces on e. Thus $h_\sigma^{e,K}$ and $h_u^{e,K}$ will depend linearly on the six quantities

$$(u_h|_{K_1})|_e,\ (\nabla u_h|_{K_1})|_e,\ (\sigma_h|_{K_1})|_e,\ (u_h|_{K_2})|_e,\ (\nabla u_h|_{K_2})|_e,\ (\sigma_h|_{K_2})|_e.$$

Exact Riemann solution for 1D conservation law

Godunov sln:

$$F_{j+1/2}^n = F(U_j^n, U_j^{n+1}) = \begin{cases} \min\limits_{U_j^n \leqslant \theta \leqslant U_{j+1}^n} f(\theta) & \text{if } U_j^n \leqslant U_{j+1}^n \\ \max\limits_{U_{j+1}^n \leqslant \theta \leqslant U_j^n} f(\theta) & \text{if } U_{j+1}^n \leqslant U_j^n. \end{cases} \tag{4.14}$$

$$(4.14)\qquad F^n_{j+1/2} = F(U^n_j, U^{n+1}_j) = \begin{cases} \min\limits_{U^n_j \leqslant \theta \leqslant U^n_{j+1}} f(\theta) & \text{if } U^n_j \leqslant U^n_{j+1} \\ \max\limits_{U^n_{j+1} \leqslant \theta \leqslant U^n_j} f(\theta) & \text{if } U^n_{j+1} \leqslant U^n_j . \end{cases}$$

$$f_x^* = \begin{cases} \min\limits_{U^L \le \theta \le U^R} f(\theta) & \text{if } U^L \le U^R \\ \max\limits_{U^R \le \theta \le U^L} f(\theta) & \text{if } U^L \ge U^R \end{cases}$$

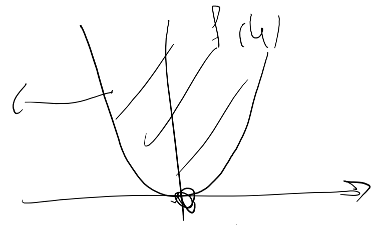
$\min(U^L, U^R) \le \theta \le \max(U^L, U^R)$

expensive

What if flux is convex (many times this is the case)

$f_x(U) = \frac{1}{2}U^2$ Burger's eqn

convex



$f(\omega)$ takes minimum

Burger's eqn $\omega = 0$

Exercise 4.1. *Computing the flux (4.14) can be complicated, since an optimization problem has to be solved. Show that in the special case where the flux function f has a single minimum at the point ω and no local maxima, the formula (4.14) can be simplified to*

$$(4.15)\qquad F^n_{j+1/2} = F(U^n_j, U^n_{j+1}) = \max\Big(f\left(\max\left(U^n_j,\ \omega\right)\right),\ f\left(\min\left(U^n_{j+1},\ \omega\right)\right)\Big).$$

$$f_x^* = \max\left(f(\max(U^L, \omega)),\ f(\min(U^R, \omega))\right)$$

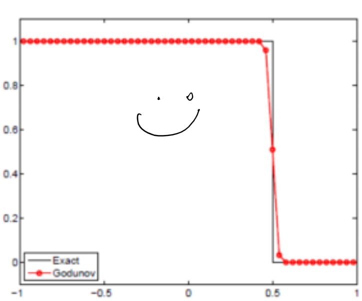
2 evaluations of f instead of a span of θ

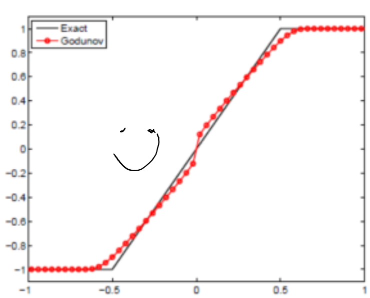
Godunov flux is exact, it's difficult to compute, but for 1D convex flux, it can take a simple form. In general, getting the Godunov flux (exact Riemann solution) is challenging.

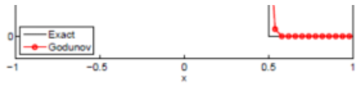
Shock example:

4.1.6. Numerical experiments. Consider Burgers' equation (3.3) with Riemann data

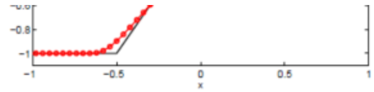
$$(4.16)\qquad U(x,0) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x > 0. \end{cases} \qquad\qquad (4.17)\qquad U(x,0) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0. \end{cases}$$







(a) Initial data (4.16) at time $t = 1$.



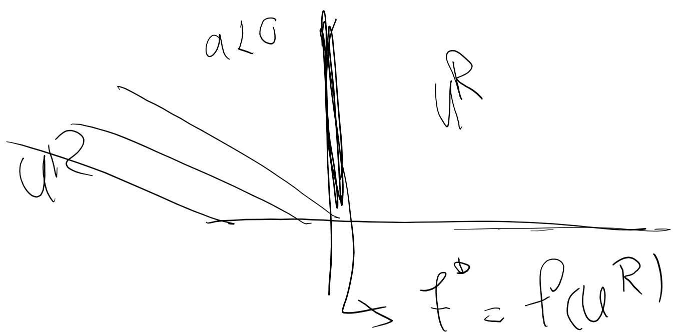
(b) Initial data (4.17) at time $t = 0.5$.

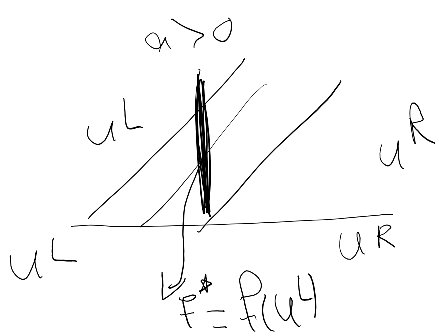
Approximate Riemann solutions:

1. Roe flux (linearlized) flux

$$\dot{u} + f(u)_{,x} = 0 \qquad \dot{u} + \underbrace{f'(u)}_{\text{speed}}\, u_{,x} = 0$$

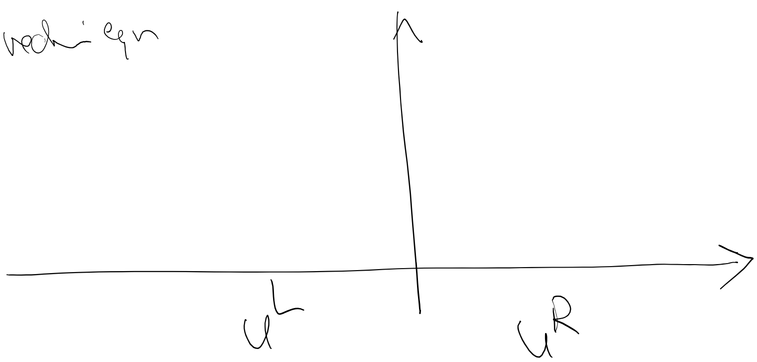
if constant $\quad \dot{u} + a\, u_{,x} = 0$





$$\dot{u} + \underbrace{f'(u)}\, u_{,x} = 0$$

if we want to treat the problem as linear advection eqn



L: $a_L = f'(u_L)$

R: $a_R = f'(u_R)$

not the same!

how I can come up with a decent choice for a unique a?

- $a = \dfrac{f'(u_L) + f'(u_R)}{2}$ (a)
- $a = f'\left(\dfrac{u_L + u_R}{2}\right)$ (b)
- $a = \begin{cases} \dfrac{f(u_R) - f(u_L)}{u_R - u_L} & u_L \neq u_R \\ f'(u) & u_L = u_R = u \end{cases}$

$\underbrace{a(u_R - u_L)}_{\text{vector}} = \underbrace{f(u_R) - f(u_L)}_{\text{vector}}$

find a that satisfies this

$\approx f'(u)$ if $u_L \approx u_R = u_0$

(

(C)

$u_R - u_L$

$f(u)$ $u_L = u_R = u$

(C)

this is the choice

$u_L \neq u_R = u_0$

more fields u $(n \times 1)$ $a_{n \times n}$ is a spatial flux matrix

(4.19) $$f(U)_x = f'(U)U_x \approx \hat{A}_{j+1/2} U_x,$$

where $\hat{A} \approx f'$ is a constant state around which the nonlinear flux function is linearized. There are many possible candidates for the linearizing state, one simple choice being

$$\hat{A}_{j+1/2} = f'\left(\frac{U_j^n + U_{j+1}^n}{2}\right)$$ (b) above

the flux of the arithmetic average of the two constant states. We will use a more sophisticated *Roe average*:

(4.20) $$\hat{A}_{j+1/2} = \begin{cases} \frac{f(U_{j+1}^n) - f(U_j^n)}{U_{j+1}^n - U_j^n} & \text{if } U_{j+1}^n \neq U_j^n \\ f'(U_j^n) & \text{if } U_{j+1}^n = U_j^n. \end{cases}$$ (C)

Roe flux for Euler's equations:
cws06_steiner_riemann.pdf

Consider again the Riemann problem

$$\mathbf{q}_t + \mathbf{f}(\mathbf{q})_x = 0\,,$$
$$\mathbf{q}(x,0) = \begin{cases} \mathbf{q}_l & \text{if} \quad x < 0\,, \\ \mathbf{q}_r & \text{if} \quad x > 0\,, \end{cases}$$

where for the x-split three-dimensional Euler equation

$$\mathbf{q} = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ E \end{pmatrix}, \qquad \mathbf{f}(\mathbf{q}) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u w \\ u(E+p) \end{pmatrix}.$$

Using the chain rule, the conservation law

$$\mathbf{q}_t + \mathbf{f}(\mathbf{q})_x = 0$$

may be written as

$$\mathbf{q}_t + \mathbf{A}(\mathbf{q})\mathbf{q}_x = 0\,, \qquad \mathbf{A}(\mathbf{q}) = \frac{\partial \mathbf{f}}{\partial \mathbf{q}}\,.$$

Roe's approach consists in replacing the Jacobian matrix $\mathbf{A}(\mathbf{q})$ by a *constant Jacobian*

$$\tilde{\mathbf{A}} = \tilde{\mathbf{A}}(\mathbf{q}_l, \mathbf{q}_r)$$ → must make sure $\tilde{A}$ has real eigenvalues

resulting in the Riemann problem for the *linear system*

$$\mathbf{q}_t + \tilde{\mathbf{A}}\mathbf{q}_x = 0\,,$$
$$\mathbf{q}(x,0) = \begin{cases} \mathbf{q}_l & \text{if} \quad x < 0\,, \\ \mathbf{q}_r & \text{if} \quad x > 0 \end{cases}$$

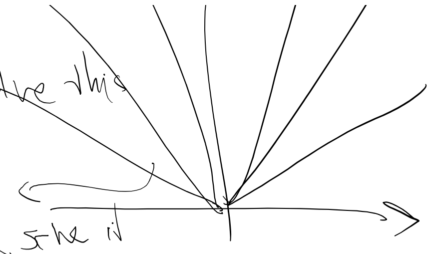
we know how to solve this

$$\mathbf{q}(x,0) = \begin{cases} \mathbf{q}_l & \text{if} \quad x < 0\,, \\ \mathbf{q}_r & \text{if} \quad x > 0\,, \end{cases}$$

which can be solved exactly.

we know how to solve this

characteristics of $\tilde{A}$ & solve it



The Roe solver (cont.)

Roe requires the constant Jacobian matrix $\tilde{\mathbf{A}} = \tilde{\mathbf{A}}(\mathbf{q}_l, \mathbf{q}_r)$ to satisfy the algebraic properties of the Jacobian $\mathbf{A}(\mathbf{q})$, i.e.,

$$\tilde{\lambda}_i = \tilde{\lambda}_i(\mathbf{q}_l, \mathbf{q}_r) \in \mathbb{R}\,, \; \forall i\,,$$

$q_L = q_R$
$$\tilde{\mathbf{A}}(\mathbf{q}, \mathbf{q}) = \mathbf{A}(\mathbf{q})\,,$$

$q_L \neq q_R$
$$\mathbf{f}(\mathbf{q}_r) - \mathbf{f}(\mathbf{q}_l) = \tilde{\mathbf{A}}(\mathbf{q}_r - \mathbf{q}_l)\,.$$

option c above

These conditions may be fullfilled with the following "Roe averagged" quantities to be used in the formulae for λ_i and $\mathbf{k}^{(i)}$ shown on the brevious pages:

$$\tilde{u} = \frac{\sqrt{\rho_l}u_l + \sqrt{\rho_r}u_r}{\sqrt{\rho_l} + \sqrt{\rho_r}}\,, \qquad \tilde{H} = \frac{\sqrt{\rho_l}H_l + \sqrt{\rho_r}H_r}{\sqrt{\rho_l} + \sqrt{\rho_r}}\,,$$

$$\tilde{v} = \frac{\sqrt{\rho_l}v_l + \sqrt{\rho_r}v_r}{\sqrt{\rho_l} + \sqrt{\rho_r}}\,, \qquad \tilde{a} = (\gamma - 1)[\tilde{H} - \frac{1}{2}\tilde{V}^2]^{\frac{1}{2}}\,,$$

$$\tilde{w} = \frac{\sqrt{\rho_l}w_l + \sqrt{\rho_r}w_r}{\sqrt{\rho_l} + \sqrt{\rho_r}}\,, \qquad \tilde{V}^2 = \tilde{u}^2 + \tilde{v}^2 + \tilde{w}^2\,.$$

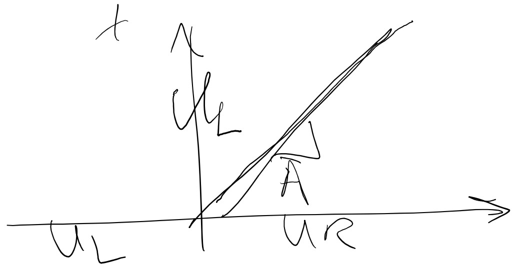
2. Consistency. All the methods we consider are consistent in the sense that, in the functional form described above,

$$h_\sigma^{e,K}(u|_{K_1}, \nabla u|_{K_1}, u|_{K_2}, \nabla u|_{K_2}) = \nabla u|_e,$$
$$h_u^{e,K}(u|_{K_1}, \nabla u|_{K_1}, u|_{K_2}, \nabla u|_{K_2}) = u|_e,$$

Continue with Roe flux but for a scalar problem

$\dot{u} + \hat{A}\, u_{,x} = 0$

$$\hat{A} = \begin{cases} \dfrac{f(u_R) - f(u_L)}{u_R - u_L} & u_R \neq u_L \\ f'(u) & u_R = u_L = u \end{cases}$$

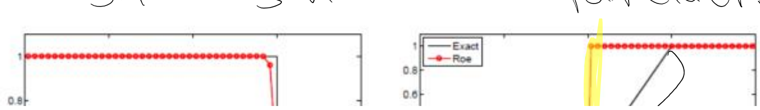


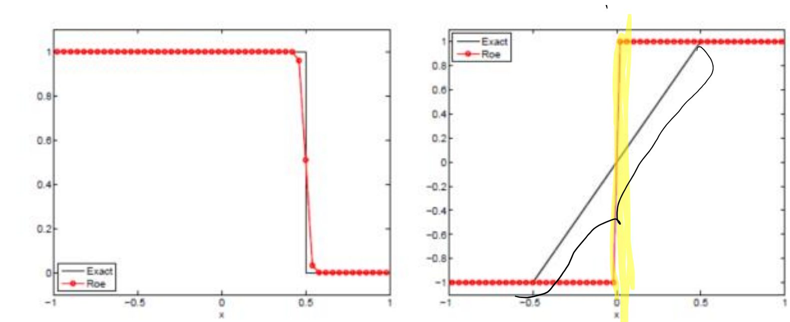
x

$\hat{A} > 0 \qquad f^* = f(u_L)$

$\hat{A} < 0 \qquad f^* = f(u_R)$

shock sin   rarefaction wave





(a) Shock solution with initial data (4.16) at $t = 1$.
(b) Rarefaction wave solution initial data (4.17) at $t = 0.5$.

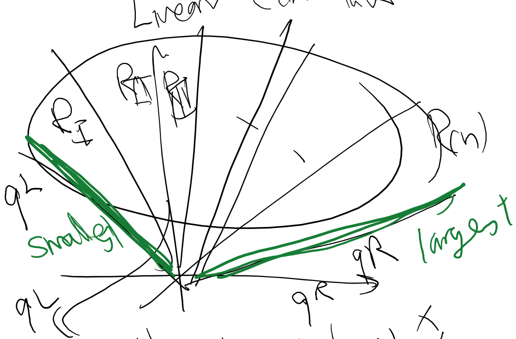
FIGURE 4.6. Approximate solutions for Burgers equation with the Roe scheme with 50 mesh points. [burgers_disc.m]

Roe flux can only model one jump between left and right states and is in capable of modeling rarefaction waves:(

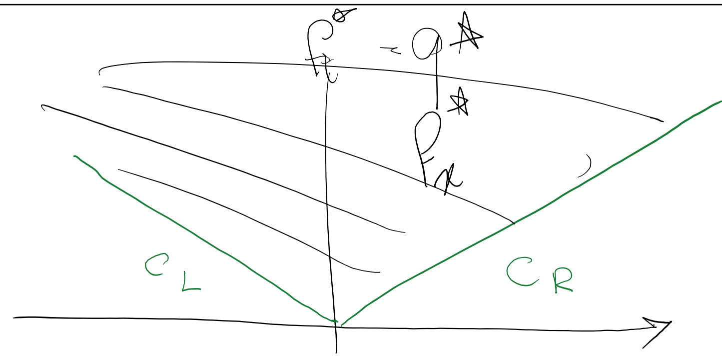
Central Schemes OR

3.1. The Riemann solver of Harten, Lax, and van Leer (HLL)

Linear Con law



all characteristics



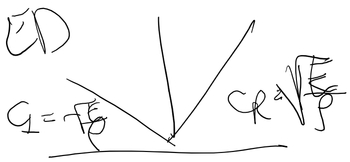
C_L minimum wave speed
C_R maximum wave speed

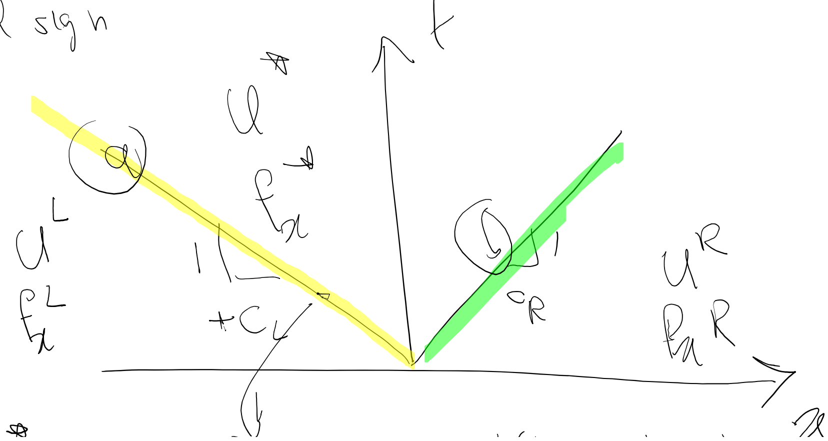
As if we are only two characteristics for a linear conservation laws

Inputs are C_L & C_R (user specified!)

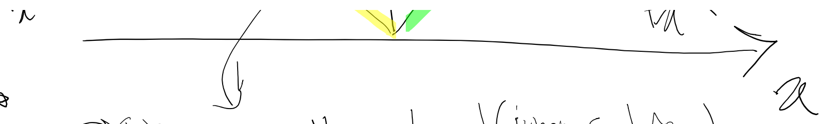
C_L is input with its actual sign

ED





$u^{L,R}$



$U^{L,R}$, $f_x^{L,R}$, C_L, C_R (chosen) $\longrightarrow$ U^*, f_x^*

(a)(b) Rankine-Hugoniot cond (jump condition)

2 eqns, 2 unknowns

$$
\begin{aligned}
\underline{a}\quad +1 \quad & C_R \times \left\{ f_x^* - f_x^L = C_L(U^* - U^L) \right. \\
+1 \quad & \underbrace{C_L}\times \left\{ f_x^R - f_x^* = C_R(U^R - U^*) \right.
\end{aligned}
$$

$$(C_R - C_L) f_x^* - C_R f_x^L + C_L f_x^R = -C_R C_L U_L + C_R C_L U_R$$

$$f_x^R - f_x^L = C_R U_R - C_L U_L + (C_L - C_R) U^*$$

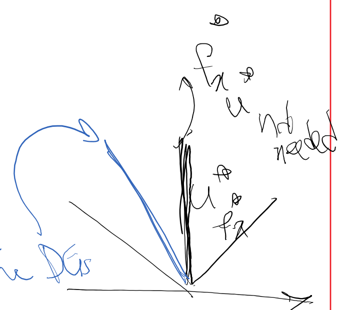
$$f_x^* = \frac{C_R f_x^L - C_L f_x^R + C_L C_R (U_R - U_L)}{C_R - C_L}$$

$$U^* = \frac{C_R U_R - C_L U_L - (f_x^R - f_x^L)}{C_R - C_L}$$

central scheme fluxes

1) U^* is it needed? yes for spacetime DGs

2) Is $f_x^* = f(U^*)$? No in general



$$\dot{U} + f(U)_{,x} = 0$$

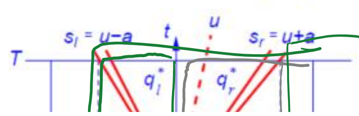
(4.25) $$f^*_{j+1/2} = \frac{s^r_{j+1/2} \overbrace{f(U^n_j)}^{f_L} - s^l_{j+1/2} \overbrace{f(U^n_{j+1})}^{f_R} + s^r_{j+1/2} s^l_{j+1/2} (\overbrace{U^n_{j+1}}^{U_R} - \overbrace{U^n_j}^{U_L})}{\underbrace{s^r_{j+1/2}}_{C_R} - \underbrace{s^l_{j+1/2}}_{C_L}}.$$

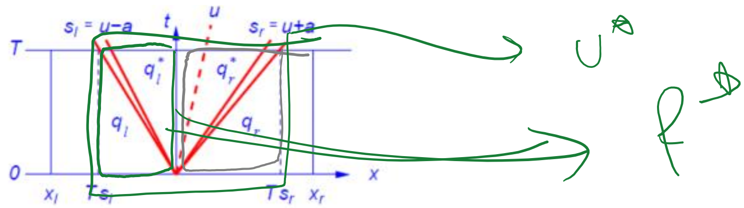
Another equivalent derivation by writing balance laws in spacetime

3.1. The Riemann solver of Harten, Lax, and van Leer (HLL)

Consider the system of one-dimensional conservation laws

$$\mathbf{q}_t + \mathbf{f}(\mathbf{q})_x = 0\,, \qquad \mathbf{q}(x,0) = \begin{cases} \mathbf{q}_l & \text{if } x < 0\,, \\ \mathbf{q}_r & \text{if } x > 0\,. \end{cases}$$





The integral form in the control volume $[x_l, x_r] \times [0, T]$ is given by:

$$\int_{x_l}^{x_r} \mathbf{q}(x,T)\mathrm{d}x = \int_{x_l}^{x_r} \mathbf{q}(x,0)\mathrm{d}x + \int_0^T \mathbf{f}(\mathbf{q}(x_l,t))\mathrm{d}t - \int_0^T \mathbf{f}(\mathbf{q}(x_r,t))\mathrm{d}t$$

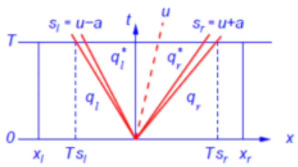
The HLL solver (cont.)

$$\begin{aligned}
\int_{x_l}^{x_r} \mathbf{q}(x,T)\,\mathrm{d}x &= \int_{x_l}^{x_r} \mathbf{q}(x,0)\,\mathrm{d}x + \int_0^T \mathbf{f}(\mathbf{q}(x_l,t))\,\mathrm{d}t - \int_0^T \mathbf{f}(\mathbf{q}(x_r,t))\,\mathrm{d}t \\
&= x_r\mathbf{q}_r - x_l\mathbf{q}_l + T(\mathbf{f}_l - \mathbf{f}_r)\,, \qquad \mathbf{f}_l = \mathbf{f}(\mathbf{q}_l)\,, \quad \mathbf{f}_r = \mathbf{f}(\mathbf{q}_r)
\end{aligned}$$

$$\begin{aligned}
\int_{x_l}^{x_r} \mathbf{q}(x,T)\,\mathrm{d}x &= \int_{x_l}^{Ts_l} \mathbf{q}(x,T)\,\mathrm{d}x + \int_{Ts_l}^{Ts_r} \mathbf{q}(x,T)\,\mathrm{d}x + \int_{Ts_r}^{x_r} \mathbf{q}(x,T)\,\mathrm{d}x \\
&= \int_{Ts_l}^{Ts_r} \mathbf{q}(x,T)\,\mathrm{d}x + (Ts_l - x_l)\mathbf{q}_l + (x_r - Ts_r)\mathbf{q}_r
\end{aligned}$$

$$\frac{1}{T(s_r - s_l)} \int_{Ts_l}^{Ts_r} \mathbf{q}(x,T)\,\mathrm{d}x := \mathbf{q}^{\mathrm{hll}} = \frac{s_r\mathbf{q}_r - s_l\mathbf{q}_l + \mathbf{f}_l - \mathbf{f}_r}{s_r - s_l}$$

The HLL solver (cont.)



Applying the integral form to the control volume $[x_l, 0] \times [0, T]$ we obtain:

$$\int_{Ts_l}^{0} \mathbf{q}(x,T)\mathrm{d}x = -Ts_l\mathbf{q}_l + T(\mathbf{f}_l - \mathbf{f}_{0l})\,,$$

where $\mathbf{f}_{0l}$ is the flux $\mathbf{f}(\mathbf{q})$ along the t-axis. Hence,

$$\mathbf{f}_{0l} = \mathbf{f}_l - s_l\mathbf{q}_l - \frac{1}{T}\int_{Ts_l}^{0} \mathbf{q}(x,T)\mathrm{d}x\,.$$

Doing the same for the control volume $[0, x_r] \times [0, T]$ leads to

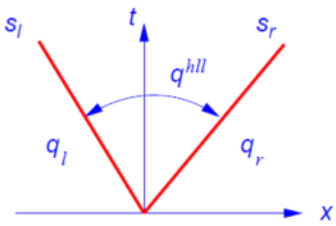
$$\mathbf{f}_{0r} = \mathbf{f}_r - s_r\mathbf{q}_r - \frac{1}{T}\int_{0}^{Ts_r} \mathbf{q}(x,T)\mathrm{d}x\,.$$

It follows that

$$\mathbf{f}_{0l} = \mathbf{f}_{0r}\,.$$

The HLL solver (cont.)

Harten, Lax, and van Leer put forward the following approximation:



$$\tilde{\mathbf{q}}(x,t) = \begin{cases} \mathbf{q}_l & \text{if} \quad \frac{x}{t} \le s_l\,, \\ \mathbf{q}^{\text{hll}} & \text{if} \quad s_l \le \frac{x}{t} \le s_r\,, \\ \mathbf{q}_r & \text{if} \quad \frac{x}{t} \ge s_r\,. \end{cases}$$

$$\mathbf{f}^{\text{hll}} = \mathbf{f}_l + s_l(\mathbf{q}^{\text{hll}} - \mathbf{q}_l) \quad \text{or}$$

$$\mathbf{f}^{\text{hll}} = \mathbf{f}_r + s_r(\mathbf{q}^{\text{hll}} - \mathbf{q}_r)$$

$$\Rightarrow \boxed{\mathbf{f}^{\text{hll}} = \frac{s_r\mathbf{f}_l - s_l\mathbf{f}_r + s_l s_r(\mathbf{q}_r - \mathbf{q}_l)}{s_r - s_l}}$$

same sln

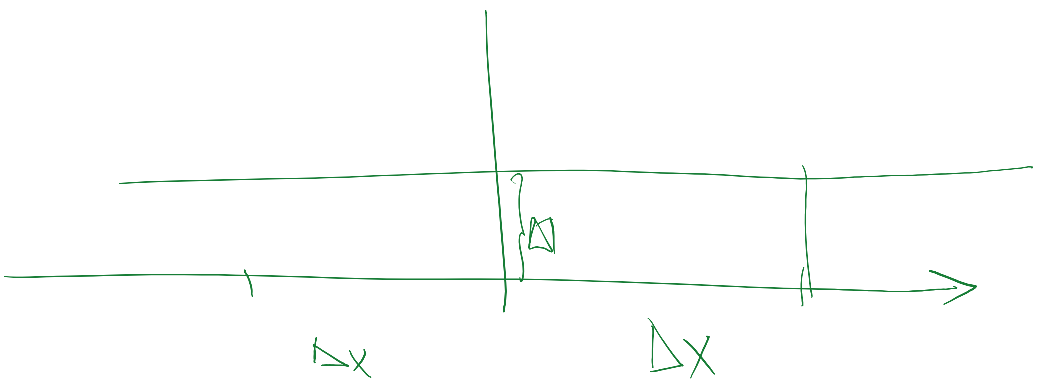
Options for cL and cR

speeds

(a) Lax – Friedrichs scheme

$c_R = \frac{\Delta x}{\Delta t}$

$c_L = -\frac{\Delta x}{\Delta t}$



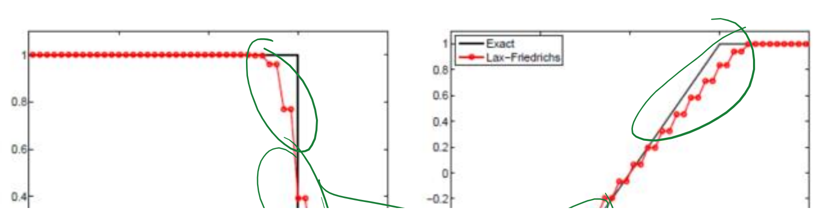
In particular, if we choose the speeds to be equal but of opposite sign, so $s^r = -s^l = s$, then (4.25) reduces to

$-c_L = c_R$

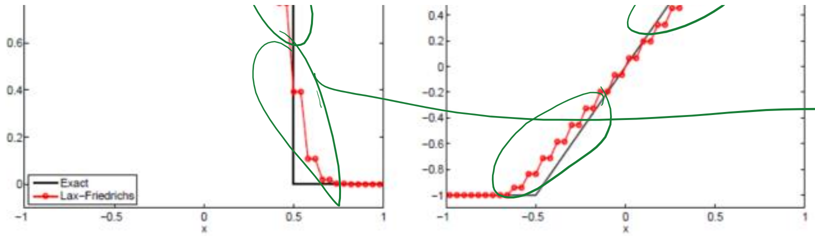
(4.26) $$f^*_{j+1/2} = \frac{f(U^n_j) + f(U^n_{j+1})}{2} - \frac{s_{j+1/2}}{2}(U^n_{j+1} - U^n_j)\,.$$

(4.29) $$F^n_{j+1/2} = F^{\text{LxF}}(U^n_j, U^n_{j+1}) = \frac{f(U^n_j) + f(U^n_{j+1})}{2} - \frac{\Delta x}{2\Delta t}(U^n_{j+1} - U^n_j)\,.$$

corresponds to numerical diffusion



→ Lax-Friedrichs



Lax-Friedrichs is too diffusive

(a) Shock solution with initial data (4.16) at $t = 1$. (b) Rarefaction wave solution with initial data (4.17) at $t = 0.5$.

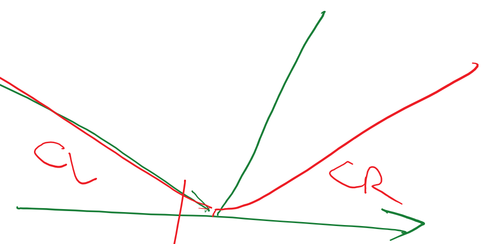
FIGURE 4.8. Approximate solution for Burgers' equation with the Lax-Friedrichs scheme with 50 mesh points. [burgers_disc.m]

4.2.4. Rusanov scheme. The Lax-Friedrichs scheme was quite diffusive around shocks. A possible explanation lies in the choice of the wave speeds (4.28). These speeds were the maximum allowed speeds and did not take into the account the speeds of propagation of the problem under consideration. A better, *locally selected*, choice of speeds is given by

$$s^r_{j+1/2} = s_{j+1/2}, \quad s^l_{j+1/2} = -s_{j+1/2}, \tag{4.30}$$

where

$$s_{j+1/2} = \max\left(|f'(U^n_j)|, \ |f'(U^n_{j+1})|\right).$$



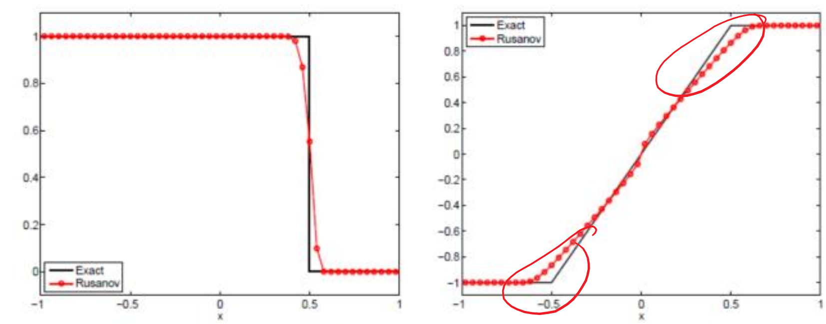
we cover domain of influence

The resulting flux (4.26), called the *Rusanov* (or *Local Lax-Friedrichs*) flux, is given by

$$\begin{aligned} F^n_{j+1/2} &= F^{\mathrm{Rus}}(U^n_j, U^n_{j+1}) \\ &= \frac{f(U^n_j) + f(U^n_{j+1})}{2} - \frac{\max\left(|f'(U^n_j)|, \ |f'(U^n_{j+1})|\right)}{2}\left(U^n_{j+1} - U^n_j\right). \end{aligned} \tag{4.31}$$

The Rusanov scheme (4.13), (4.31) leads to a considerable improvement in results over the Lax-Friedrichs scheme, as shown in Figure 4.9.

$$\begin{aligned} F^n_{j+1/2} &= F^{\mathrm{Rus}}(U^n_j, U^n_{j+1}) \\ &= \frac{f(U^n_j) + f(U^n_{j+1})}{2} - \frac{\max\left(|f'(U^n_j)|, \ |f'(U^n_{j+1})|\right)}{2}\left(U^n_{j+1} - U^n_j\right). \end{aligned} \tag{4.31}$$



Much less diffusive than Lax-F method

(a) Shock solution with initial data (4.16) at $t = 1$. (b) Rarefaction wave solution with initial data (4.17) at $t = 0.5$.

FIGURE 4.9. Approximate solution for Burgers' equation with the Rusanov scheme using 50 mesh points. [burgers_disc.m]