DG2020/04/20

Monday, April 20, 2020 11:43 AM

From last time:

$$f_x^* = \frac{c_R f_x^L - c_L f_x^R + c_L c_R (u_R - u_L)}{c_R - c_L}$$

$$u^* = \frac{c_R u_R - c_L u_L - (f_x^R - f_x^L)}{c_R - c_L}$$

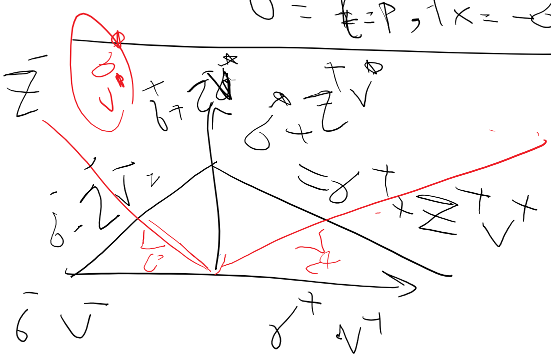
(1)

Are these fluxes correct for linear conservation laws?

elastodynamics $\dot{p} - \sigma_{,x} = 0$

$U = p, \; f_x = -\sigma$

we have already solved the exact Riemann sln for this linear system.



2eqn 2unknowns

$$\left.\begin{aligned} \sigma^* - Z^- v^* &= \sigma^- - Z^- v^- \\ \sigma^* + Z^+ v^* &= \sigma^+ + Z^+ v^+ \end{aligned}\right\} \rightarrow$$

$$\boxed{\begin{aligned} v^* &= \frac{(\sigma^+ - \sigma^-) + (Z^+ v^+ + Z^- v^-)}{Z^- + Z^+} \\ \sigma^* &= \frac{(Z^- \sigma^+ + Z^+ \sigma^-) + Z^- Z^+ (v^+ - v^-)}{Z^- + Z^+} \end{aligned}}$$

(2)

$Z^{\pm} = c^{\pm} \rho^{\pm}$

$p = \rho v$

$\sigma^* = c^- \rho^- \sigma^- \ldots$

Under what conditions we have a unique value for p*?

$p^{*-} = \rho^- v^*$ $\quad p^{*+} = \rho^+ v^*$

$\rho^- = \rho^+ \longrightarrow c^-/c^+ = Z^-/Z^+ = \dfrac{c^- \rho^-}{c^+ \rho^+}$

$$p^* = \frac{(\sigma^+ - \sigma^-) + (c^+ p^+ + c^- p^-)}{c^- + c^+}$$

$$\sigma^* = \frac{c^- \sigma^+ + c^+ \sigma^- + c^- c^+ (p^+ - p^-)}{c^- + c^+}$$

exact for $\rho^- = \rho^+$

$c^- + c^+$

$$f_x^* = \frac{c_R f_x^L - c_L f_x^R + c_L c_R (u_R - u_L)}{c_R - c_L} \quad (-\sigma^+ + \sigma^-)$$

$c^+ p^- + c^- p^+$

$$u^* = \frac{c_R u_R - c_L u_L - (f_x^R - f_x^L)}{\underbrace{c_R}_{c^+} - \underbrace{c_L}_{c^-}}$$

p^*

approximate (HLL)

$c_R = c^+$

$\boxed{c_L = c^-}$

$u = p$

$f_x = -\sigma$

As we can see, HLL flux is only exact for linear cons. laws when there is only one sector in the middle (i.e. 2 eigenvalues <-> 2x2 spatial flux matrix) with same material properties on the two sides (or slightly more general)

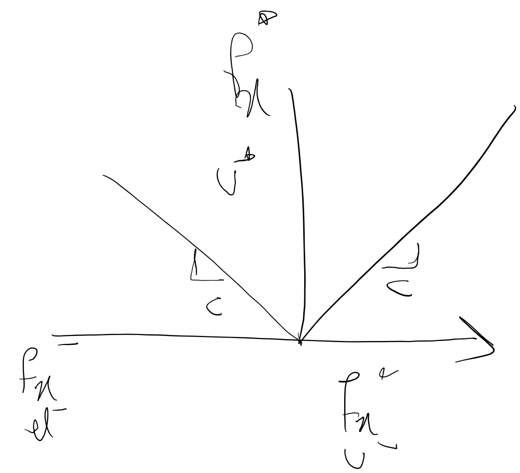
HLL

Most often cR = -cL = c

$$f_x^* = \frac{c_R f_x^L - c_L f_x^R + c_L c_R (u_R - u_L)}{c_R - c_L}$$

$$u^* = \frac{c_R u_R - c_L u_L - (f_x^R - f_x^L)}{c_R - c_L}$$

$$\boxed{\begin{aligned} f_x^* &= \left(\frac{f_x^- + f_x^+}{2}\right) - \frac{c}{2}(u^+ - u^-) \\ u^* &= \left(\frac{u^- + u^+}{2}\right) - \frac{1}{2c}(f_x^+ - f_x^-) \end{aligned}}$$



$\boxed{\text{L.F.} \quad c = \frac{\Delta x}{\Delta t}}$ can be too too b

Rusanov $\quad c = \max(f'^-, f'^+)$

local L.F.

$\dot{u} + f_{x,x} = 0$

$\dot{u} + \frac{\partial f_x}{\partial u} u_{,x} = 0$

f'

There are many cases that we may want to use different c_L, c_R

Adv-Diffusion eqn

$$\begin{cases} C\dot{c} - \nabla\cdot q = S \\ \underbrace{\tau\dot{q}}_{} + k\nabla c = -q \end{cases}$$

$\dot{c}$ ↓ concentration

Diffusion eqn

$S = -\beta c$ reaction

$\tau = 0$ (parabolic) / $\tau > 0$ hyperbolic

+ Advection $\quad C\dot{c} \to C(\underbrace{\cancel{\dot{c}} + \nabla\cdot vc}_{})$ ↓ advection speed from fluid

$$\tau\dot{q} \to \tau(\dot{q} + \nabla\cdot vq)$$

$$C(\dot{c} + \nabla\cdot(vc)) + \nabla\cdot q = S$$
$$\tau(\dot{q} + \nabla\cdot(vq)) + k\nabla c = -q$$

$\longrightarrow$ C constant

$$\begin{cases} C\dot{c} + \nabla\cdot(Ccv + q) = S \\ \tau\dot{q} + \nabla\cdot(\tau vq + kc) = 0 \end{cases}$$

wave speeds $\tilde{q}$

$$\dot{c} + \nabla\cdot(cv + q/C) = S$$
$$\dot{q} + \nabla\cdot\left(vq + \frac{k}{\tau}c\right) = 0$$

flux

spatial flux $\longleftarrow$ $A = \begin{bmatrix} v & \frac{1}{C} \\ \frac{k}{\tau} & v \end{bmatrix}$

eigenvalues are the wave speeds

$\det(A - \lambda I) = 0 \quad \det\begin{bmatrix} v-\lambda & \frac{1}{C} \\ \frac{k}{\tau} & v-\lambda \end{bmatrix} = 0 \quad (\lambda - v)^2 - \frac{k}{C\tau} = 0$

$\lambda = v + \sqrt{k}$

use

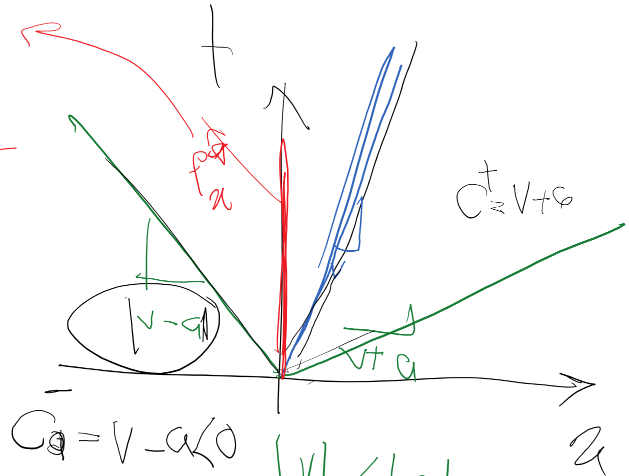
$$\lambda = \underbrace{V}_{\text{advective speed from fluid given}} \pm \underbrace{\sqrt{\frac{K}{C\Sigma}}}_{\text{diffusive speed}}$$

advective speed

from fluid

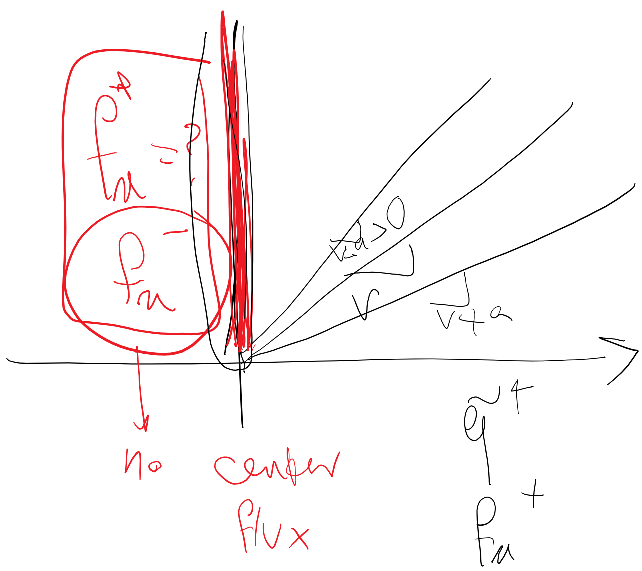
given

diffusive speed



$|V| \leq |a|$

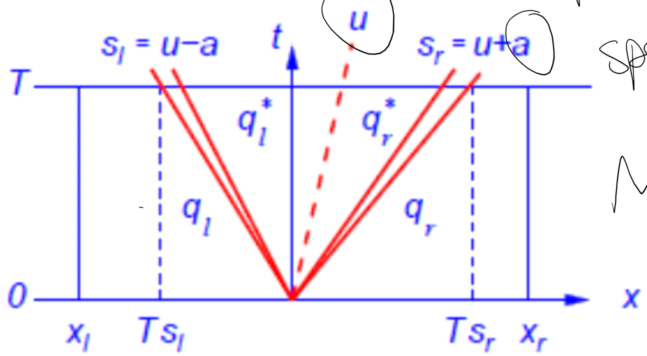
Diffusive dominant transport



$v > 0$

$|V| > |a|$

Euler's equation:

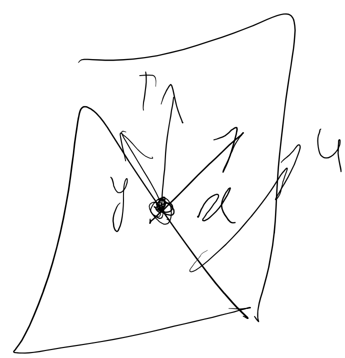


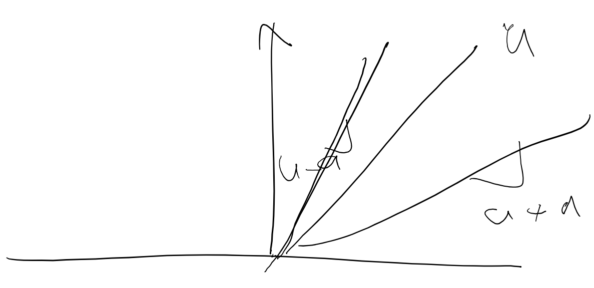
fluid speed along x

speed of sound

$M = \frac{u}{a} < 1$

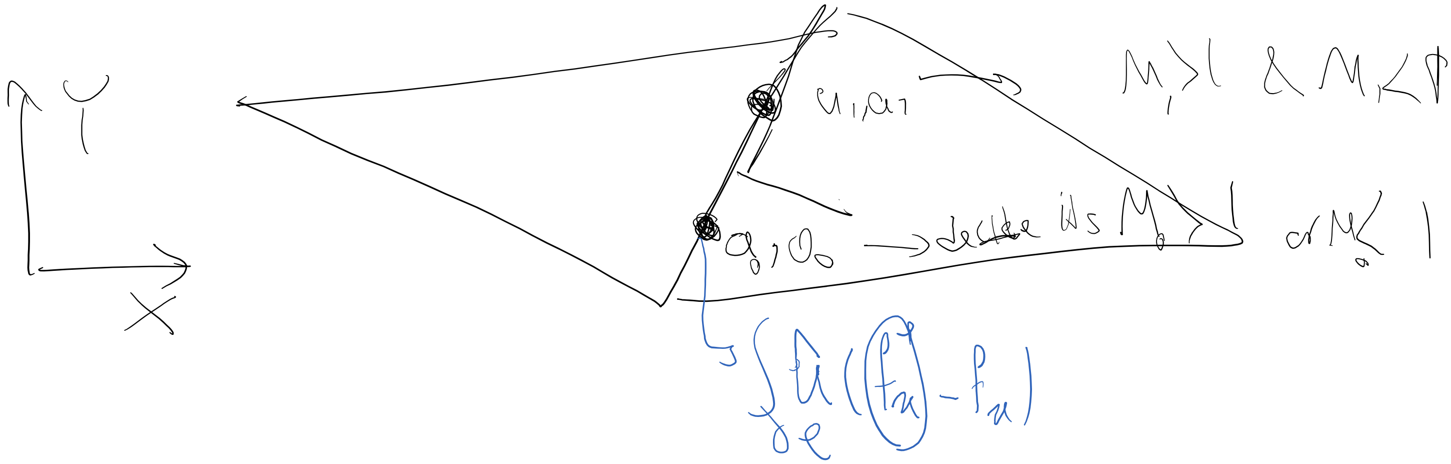
subsonic





center flux

$M > 1$



Read

4.2.5. Engquist-Osher scheme. A related scheme is the Engquist-Osher scheme, which has flux

$$
\begin{aligned}
F^n_{j+1/2} &= F^{\mathrm{EO}}(U^n_j, U^n_{j+1}) \\
&= \frac{f(U^n_j) + f(U^n_{j+1})}{2} - \frac{1}{2}\int_{U^n_j}^{U^n_{j+1}} |f'(\theta)|\, d\theta.
\end{aligned}
\tag{4.32}
$$

For convex spatial flux we can get rid of integral (the same way that in Godunov flux we could get rid of min operator) to get:

Although it is difficult to write the Engquist-Osher flux as an approximate Riemann solver, it shares several features of approximate Riemann solvers. When the flux function has a single minimum at a point ω and no maxima (which is the case for most convex functions), the Engquist-Osher flux can be explicitly computed as

$$
F^{\mathrm{EO}}(U^n_j, U^n_{j+1}) = f\left(\max\left(U^n_j, \omega\right)\right) + f\left(\min\left(U^n_{j+1}, \omega\right)\right) - f(\omega). \tag{4.33}
$$

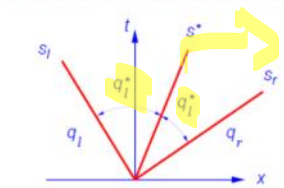
Compare this with exact Godunov flux

$$
F^n_{j+1/2} = F(U^n_j, U^n_{j+1}) = \max\Big(f\left(\max\left(U^n_j,\ \omega\right)\right),\ f\left(\min\left(U^n_{j+1},\ \omega\right)\right)\Big). \tag{4.15}
$$

For fluids, read HLLC
cws06_steiner_riemann.pdf

Extend HLL by allowing a contact discontinuity for obtaining approximate solution:

The HLLC scheme is a modification of the HLL scheme in which the missing *contact and shear waves are restored*.

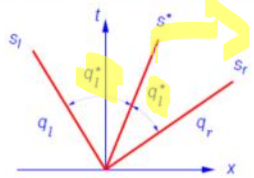


Contact discontinuity

$$
\tilde{\mathbf{q}}(x,t) = \begin{cases}
\mathbf{q}_l & \text{if} \quad \frac{x}{t} \le s_l\,, \\
\mathbf{q}_l^* & \text{if} \quad s_l \le \frac{x}{t} \le s^*\,, \\
\mathbf{q}_r^* & \text{if} \quad s^* \le \frac{x}{t} \le s_r\,, \\
\mathbf{q}_r & \text{if} \quad \frac{x}{t} \ge s_r\,.
\end{cases}
$$

Integrating over appropriate control volumes, or more directly, by applying the Rankine-Hugoniot Conditions across each wave, we obtain

The HLLC scheme is a modification of the HLL scheme in which the missing *contact and shear waves are restored*.



Contact discontinuity

$$\tilde{\mathbf{q}}(x,t) = \begin{cases} \mathbf{q}_l & \text{if} \quad \frac{x}{t} \le s_l \,, \\ \mathbf{q}_l^* & \text{if} \quad s_l \le \frac{x}{t} \le s^* \,, \\ \mathbf{q}_r^* & \text{if} \quad s^* \le \frac{x}{t} \le s_r \,, \\ \mathbf{q}_r & \text{if} \quad \frac{x}{t} \ge s_r \,. \end{cases}$$

Integrating over appropriate control volumes, or more directly, by applying the Rankine-Hugoniot Conditions across each wave, we obtain

$$\begin{aligned} \mathbf{f}_l^* &= \mathbf{f}_l + s_l(\mathbf{q}_l^* - \mathbf{q}_l) \,, \\ \mathbf{f}_r^* &= \mathbf{f}_l^* + s^*(\mathbf{q}_r^* - \mathbf{q}_l^*) \,, \\ \mathbf{f}_r^* &= \mathbf{f}_r + s_r(\mathbf{q}_r^* - \mathbf{q}_r) \,. \end{aligned}$$

The intermediate states $\mathbf{q}_l^*$ and $\mathbf{q}_r^*$ can be derived from

$$\begin{aligned} \mathbf{f}_l^* &= \mathbf{f}_l + s_l(\mathbf{q}_l^* - \mathbf{q}_l) \,, \\ \mathbf{f}_r^* &= \mathbf{f}_l^* + s^*(\mathbf{q}_r^* - \mathbf{q}_l^*) \,, \\ \mathbf{f}_r^* &= \mathbf{f}_r + s_r(\mathbf{q}_r^* - \mathbf{q}_r) \,, \end{aligned} \qquad \begin{aligned} & u_l^* = u_r^* = u^* \,, \\ & p_l^* = p_r^* = p^* \,, \\ & v_l^* = v_l \,, \quad v_r^* = v_r \,, \\ & w_l^* = w_l \,, \quad w_r^* = w_r \,, \\ & s^* = u^* \,, \end{aligned}$$

tangential velocities are equal to l, r places

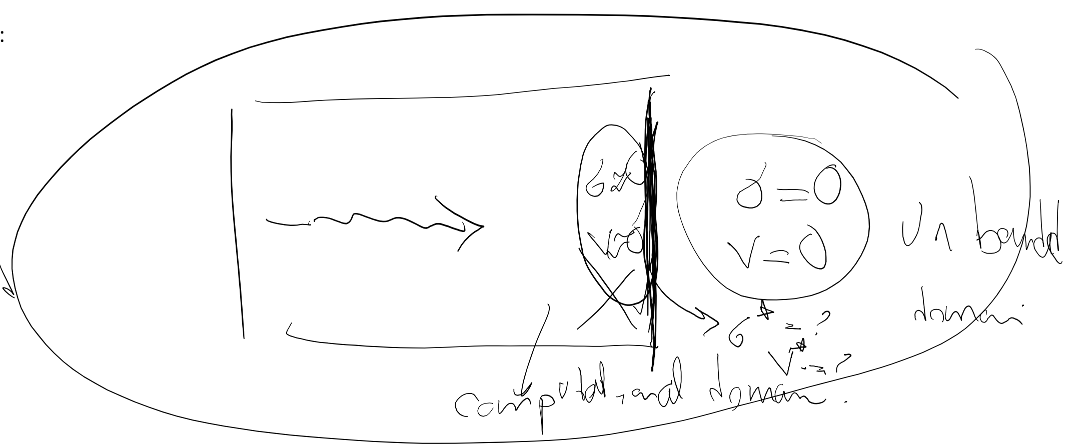
$$\mathbf{q}_k^* = \rho_k \left(\frac{s_k - u_k}{s_k - s^*} \right) \begin{bmatrix} 1 \\ s^* \\ v_k \\ w_k \\ \frac{E_k}{\rho_k} + (s^* - u_k)[s^* + \frac{p_k}{\rho_k(s_k - u_k)} \end{bmatrix} , \qquad k = l, r$$

Transmitting BC and relation to Riemann solution:

Dirichlet

$\sigma^* = \sigma$

$\boxed{V^* = \bar{V}}$ → specified



Neumann

$\bar{t}$

$\boxed{\sigma_n^* = \bar{t}}$

$V^* = V$

transmitting BC

transmitting BC

$\sigma^* = \bar{\sigma} = 0$

$v^* = \bar{v} = 0$

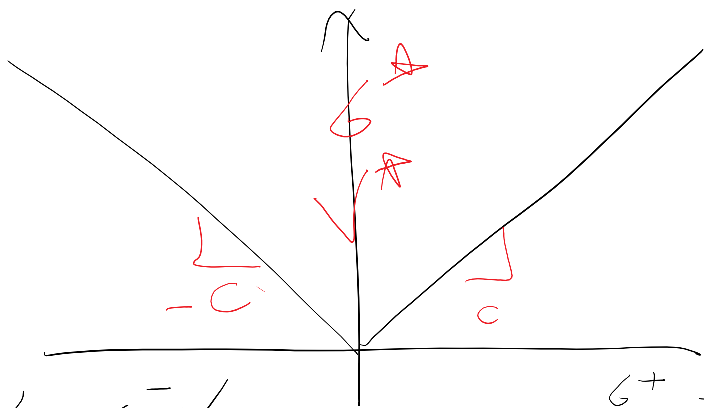
Res $\sigma^* - \bar{\sigma} \to 0$

Res $v^* - \bar{v} \to 0$

$h \to 0$

Apply the Riemann sln

interior trace



$\sigma = \sigma^-$ ✓

$v = v^-$ ✓

$\sigma^+ = 0$

$v^+ = 0$

exterior are in fact zero!

$\sigma^+ = 0 \quad v^+ = 0$

$\sigma^- = \sigma \quad v^- = v$

$Z^- = Z^+ = Z$

$$v^* = \frac{(\sigma^+ - \sigma^-) + (Z^+ v^+ + Z^- v^-)}{Z^- + Z^+}$$

$$\sigma^* = \frac{(Z^- \sigma^+ + Z^+ \sigma^-) + Z^- Z^+ (v^+ - v^-)}{Z^- + Z^+}$$

$$\boxed{\begin{aligned} v^* &= \frac{-1}{2Z}(\sigma - Zv) \\ \sigma^* &= \frac{1}{2}(\sigma - Zv) \end{aligned}}$$

Silver-Muller or Sommerfeld absorbing BC

Left going wave

$\sigma - Zv = $ constant

$\sigma - Zv = 0 - Z0$

$\sigma = 0, \; v = 0$

BC's & DG methods

Dirichlet

$\sigma^- = \sigma$ $\qquad \sigma^+ = \sigma$

$v^- = v$ $\qquad v^+ = \bar{v}$

optional

$\rightarrow \sigma^{\oplus} = \sigma$

$v^{\oplus} = \bar{v}$

skip the Riemann if wanted

Neumann

$\sigma^- = \sigma$ $\qquad \sigma_n^+ = \bar{\sigma}_n$

$v^- = v$ $\qquad v^+ = v$

Transmitting BC

$\sigma^- = \sigma$ $\qquad \sigma^+ = 0$

$v^- = v$ $\qquad v^+ = 0$

must

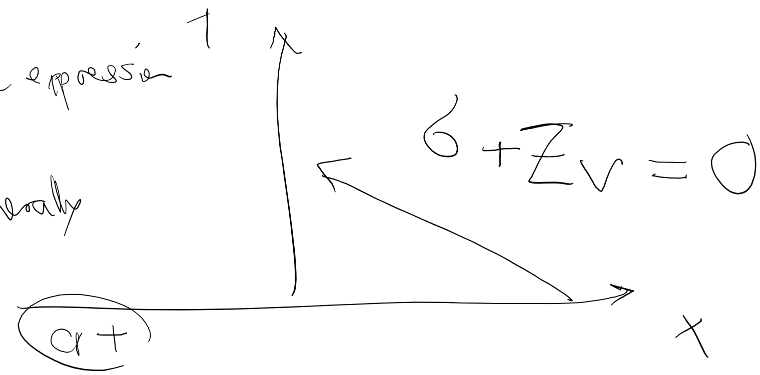
Apply Riemann sln to get $\sigma_n^{\oplus}, v^{\oplus}$ for vertical face

Other expressions of Sommerfeld BC

$$\boxed{\sigma + Zv = 0}$$

other expression

For elliptic PDEs we generally look for steady state or harmonic state sln.



$$\hat{f}(t) = \frac{1}{} \int^{\infty}_{0} e^{-i\omega t} f(t)\,dt$$

$$\hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\omega t} f(t)\, dt$$

$$\sigma + Z v = 0 \longrightarrow \hat{\sigma} . n + Z \hat{v} = 0$$

$$v = \dot{u} \longrightarrow \hat{v} = i\omega \hat{u}$$

$$\boxed{\hat{\sigma} n + i\omega Z_n \hat{u} = 0}$$

$$\sigma = \underbrace{C}_{\text{elasticity tensor}} \varepsilon = C \nabla u$$

$$\hat{\sigma} = C \nabla \hat{u}$$

$$C \nabla \hat{u} . n + i\omega Z_n \hat{u} = 0$$

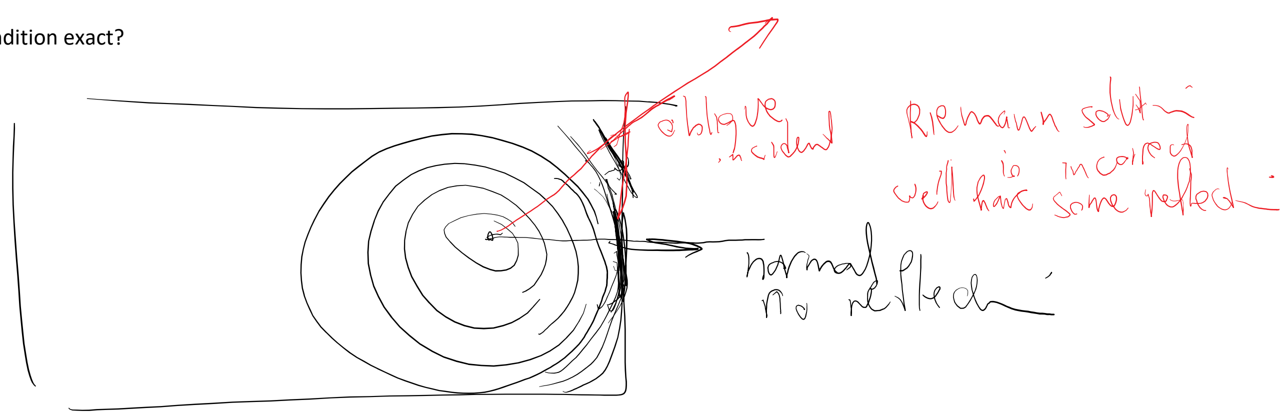
if isotropic & ... longitudinal

$$\underbrace{c^2 \rho}_{\text{wave speed}} \nabla \hat{u} . n + i\omega c \rho \hat{u} = 0$$

$$\boxed{c \nabla \hat{u} . n + i\omega \hat{u} = 0}$$

where elliptic PDE solves $\hat{u}$

Is this boundary condition exact?



Sommerfeld / Silver-Müller 1D exact

PML is better but more cumbersome!