2017/08/31

Thursday, August 31, 2017 8:35 AM

Last point about determinant:

If we have even number of permutations in the rows (columns) of a matrix the determinant doesn't change. With even number permutations determinant is multiplied by -1.

$$B = \begin{bmatrix} A_{21} & A_{22} & A_{23} \\ A_{11} & A_{12} & A_{13} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

row 1 & 3

$$\det B = \epsilon_{ijk} B_{1i} B_{2j} B_{3k}$$

$$\begin{cases} B_{1i} = A_{2i} \\ B_{2j} = A_{1j} \\ B_{3k} = A_{3k} \end{cases}$$

$$= \epsilon_{ijk} A_{2i} A_{1j} B_{3k}$$

$$= \epsilon_{ijk} A_{1j} A_{2i} A_{3k}$$

1 permutation

$$= -\epsilon_{jik} A_{1j} A_{2i} A_{3k}$$

$$= -\det A$$

Can generalize this to reach the statement at the top.

- Also you are going to find the expression for A inverse:

$$\mathbf{A}\mathbf{x} = \mathbf{b} \; (x_r = A^{-1}_{rk} b_k) \quad \Rightarrow x_r = \frac{1}{2 \det A} \epsilon_{ijk} \epsilon_{pqr} A_{ip} A_{jq} b_k$$

Two more examples with indicial notation:

$$z = Ax + By$$

$$z_i = A_{ij} x_j + B_{ij} y_j$$

$$y = Cu \qquad y_j = C_{ji} u_i$$

should have used k

$$z_i = A_{ij} x_j + B_{ij} C_{ji} u_i$$

k

i is repeated 3 times

$$z_i = A_{ij} x_j + B_{ij} C_{jk} u_k$$

$$Q = x^T A x$$

want to minimize Q

$$Q = x_i A_{ij} x_j$$

$$\frac{\partial Q}{\partial x_j} = \frac{\partial x_i A_{ij} x_j}{\partial x_j} = x_i A_{ij}$$

j is repeated 3 times

WRONG

$$\frac{\partial Q}{\partial x_k} = \frac{\partial x_i A_{ij} x_j}{\partial x_k} = \overset{\delta_{ik}}{\frac{\partial x_i}{\partial x_k}} A_{ij} x_j + x_i A_{ij} \underset{\delta_{jk}}{\frac{\partial x_j}{\partial x_k}}$$

replace i with k

$$= \delta_{ik} A_{ij} x_j + x_i A_{ij} \delta_{jk}$$

$$= A_{kj} x_j + \underbrace{x_i A_{ik}}$$

change these two i's to j's

$$= A_{kj} x_j + A_{jk} x_j$$

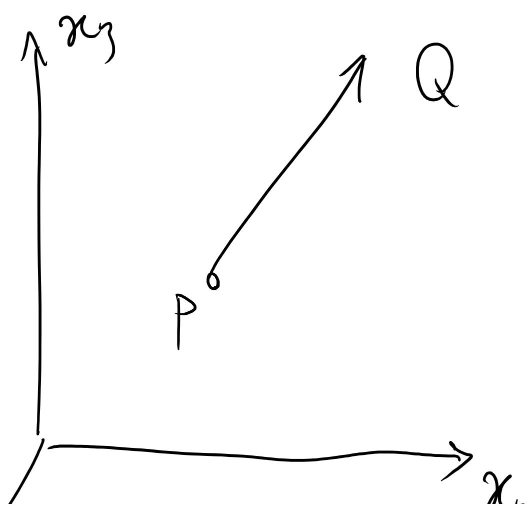
$$= \underbrace{(A_{kj} + A_{jk})}_{2 \text{ time Sym } A} x_j$$

$$\frac{\partial Q}{\partial x} = 2 (\text{Sym } A) x$$

New topic:

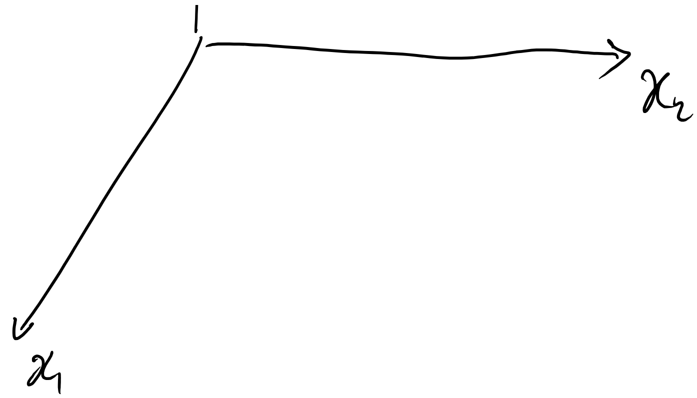
Vector Spaces

3D Euclidean space



$\overrightarrow{PQ}$ is a vector

properties of $\vec{PQ}$:



1. It has length
2. and direction

(3.) absolute base position (P)

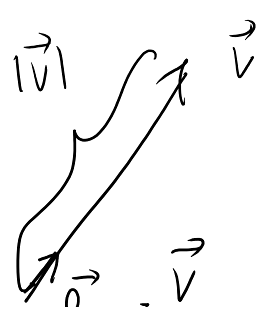
3rd property mentions what is the base point of the vector. Many times we don't care about the base point of a vector.

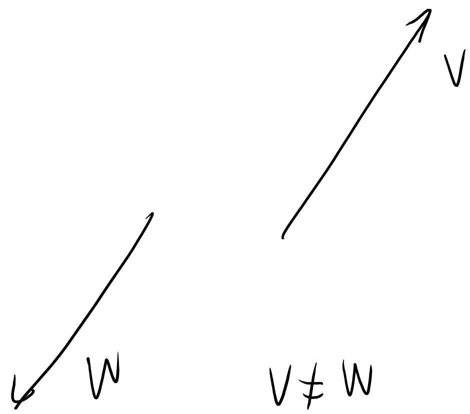
If property 3 is important and needed we call the vector a "bound vector"

Often we don't care about the base of a vector. In that case the two vectors v and w are equal iff:

$|v| = |w|$ magnitudes are equal

$\vec{e}_v = \vec{e}_w$ same direction





$V \neq W$

$\vec{e}_v = \frac{\vec{v}}{|v|}$

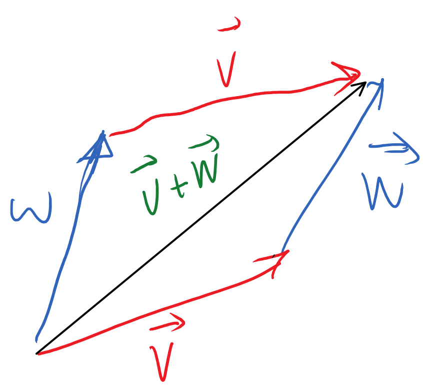
$\left(|v| = |w| \right.$

$\left. \vec{e}_v = -\vec{e}_w \right)$

Operations on vectors:

1. Vector addition

$\vec{v}$ & $\vec{w}$: want to define $\vec{v} + \vec{w}$



$\vec{v} + \vec{w} = \vec{w} + \vec{v}$

2. Scalar product

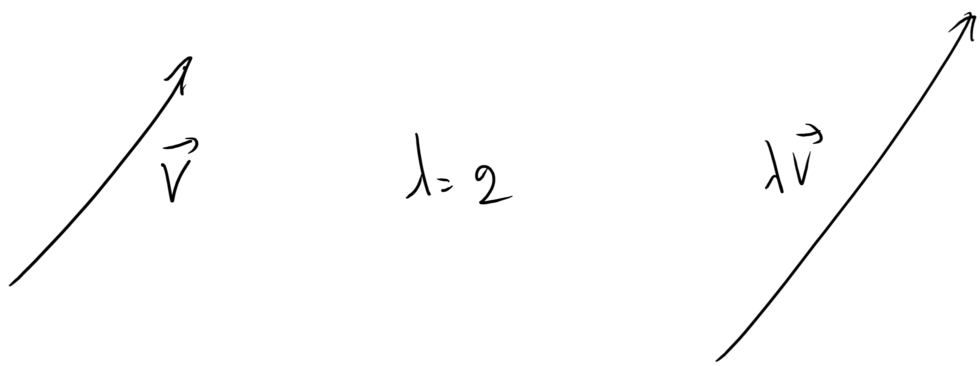
$\lambda \in \mathbb{R}$ define $\lambda \vec{v}$

$\vec{v}$ a vector

$\lambda \vec{v}$ is a vector with magnitude $|\lambda||\vec{v}|$

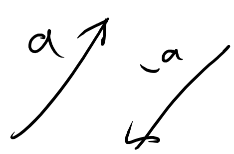
and direction $\begin{cases} e_{\vec{v}} & \lambda > 0 \\ -e_{\vec{v}} & \lambda < 0 \end{cases}$

and $\lambda \vec{v} = \vec{0}$ if $\lambda = 0$



$\lambda = -\frac{1}{2}$

$-\vec{a} := (-1)\vec{a}$



$$\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$$

Properties of vectors:

A. Addition properties

$$\begin{cases} \vec{a} + \vec{b} = \vec{b} + \vec{a} & \text{commutative} \\ \vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c} & \text{assocative} \\ \vec{a} + 0 = 0 + \vec{a} = \vec{a} & \end{cases}$$

0: is "zero" member

B. Scalar product properties

$\lambda, \mu \in \mathbb{R}$ a, b : a vector

1) $(\lambda\mu)\, a$ $\lambda(\mu a)$ scalar product property

2) $(\lambda + \mu)\vec{a} = \lambda\vec{a} + \mu\vec{a}$ distributive w.r.t. scalar addition

3) $\lambda(\vec{a} + \vec{b}) = \lambda\vec{a} + \lambda\vec{b}$ distributive w.r.t. vector addition

4) $1 . \vec{a} = \vec{a}$

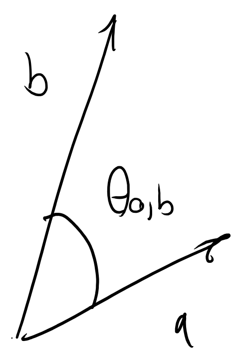
Later any space that satisfies properties above will be called a vector space.

Inner product:

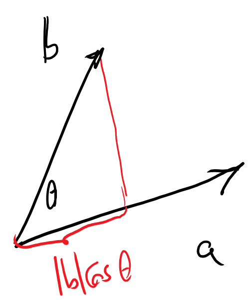
$\vec{a}$ and $\vec{b}$ are two vectors

$\vec{a} \cdot \vec{b}$ is a real number

$\vec{a} \cdot \vec{b} = |a|\,|b| \cos \theta_{a,b}$



$|b| \cos \theta$



$$a \cdot b = |a| \, |\text{Projection of } \mathbf{b} \text{ on } a|$$

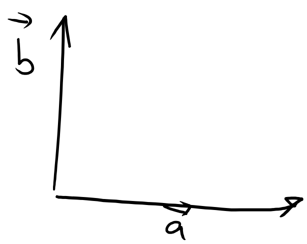
$$= |a| \, |P_{\vec a}(\vec b)|$$

Interesting cases for inner product:

1) $\theta = 0$

$$\vec a \cdot \vec b = |a||\vec b| \cos\theta = |\vec a||\vec b|$$

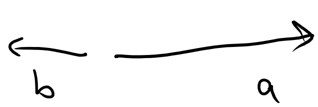
2) $\theta = \frac{\pi}{2}$



$$\vec a \cdot \vec b = |\vec a||\vec b| \overbrace{\cos\left(\tfrac{\pi}{2}\right)}^{=0} = 0$$

$\vec a$ & $\vec b$ are normal $\qquad \vec a \cdot \vec b = 0 \qquad \theta_{a,b} = \pm 90^\circ$

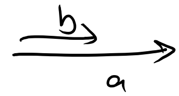
3) $\theta = \pi$



$$\vec a \cdot \vec b = |\vec a||\vec b| \cos\pi = -|\vec a||\vec b|$$

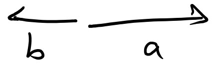
If we have two vectors a and b, when is a.b maximum?

$\theta_{a,b} = 0$ Max $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|$



And we get the minimum value when

$\theta = \pi$



$\vec{a} \cdot \vec{b} = -|\vec{a}||\vec{b}|$

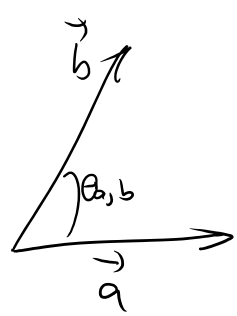
BTW if $\vec{a} = 0$ or $\vec{b} = 0$ $\vec{a} \cdot \vec{b} = 0$

(this is the case $\theta_{\vec{a},\vec{b}}$ cannot be defined)

Properties of inner product:

1. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ commutative

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta_{a,b}$$
$$= |\vec{b}||\vec{a}| \cos \theta_{b,a}$$
$$= \vec{b} \cdot \vec{a}$$

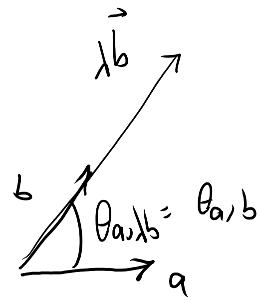


2. $\vec{a} \cdot (\lambda \vec{b}) = \lambda(\vec{a} \cdot \vec{b})$ λ is a scalar

three cases for λ

- $\lambda = 0$ $\left.\begin{array}{l} \vec{a} \cdot (0\vec{b}) = \vec{a} \cdot \vec{0} = 0 \\ 0(\vec{a} \cdot \vec{b}) = 0 \end{array}\right\} \vec{a} \cdot (\lambda\vec{b}) = \lambda(\vec{a} \cdot \vec{b})$

- $\lambda > 0$



$\vec{a} \cdot (\lambda \vec{b}) =$
$|\vec{a}|\,|\lambda\vec{b}| \cos\theta_{a,\lambda\vec{b}}$
$= |\vec{a}|\,|\lambda|\,|\vec{b}| \cos\theta_{a,b}$

$= \underbrace{|\lambda|}_{\lambda\ (\lambda > 0)} \underbrace{\{|\vec{a}|\,|\vec{b}| \cos\theta_{a,b}\}}_{a \cdot b}$

$= \lambda(\vec{a} \cdot \vec{b})$

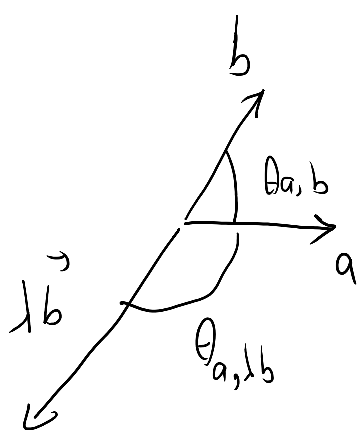
$\vec{a} \cdot (\lambda\vec{b}) = \lambda(\vec{a} \cdot \vec{b})$
$\lambda > 0$

- $\lambda < 0$

$\theta_{a,\lambda b} = \pi - \theta_{a,b}$

$\vec{a} \cdot \lambda\vec{b} =$

$|a|\,|\lambda b| \cos\theta_{a,\lambda b}$



$|a||\lambda b| \cos\theta_{a,\lambda b}$ ↙ $\vec{a},\lambda b$

$|a||\lambda||b| \cos(\pi - \theta_{a,b})$

$= |\lambda||a||b| \left(\cos\theta_{a,b}\right)$

$= \underbrace{\left(-|\lambda|\right)}_{\lambda \text{ for } \lambda<0} \underbrace{|a||b|\cos\theta_{a,b}}_{\vec{a}\cdot\vec{b}} = \lambda\, \vec{a}\cdot\vec{b}$

$\vec{a}\cdot(\lambda b) = \lambda\, \vec{a}\cdot\vec{b}$ all three cases

3) $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$

distributive property of inner product w.r.t. vector addition

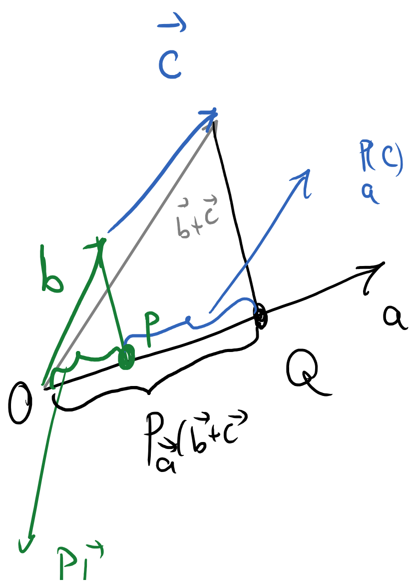
Proof:

$\vec{a}\cdot(\vec{b}+\vec{c}) =$

$|a|\, P_{\vec{a}}(\vec{b}+\vec{c})$

$= |a|\, |OQ|$

$\vec{a}\cdot\vec{b} = |a|\, P_{\vec{a}}(\vec{b}) =$



$\vec{a}\cdot\vec{b} = |a|\, P_{\vec{a}}(\vec{b}) =$ $\qquad P_{\vec{a}}\vec{b}$

$|a|\;|OP|$

$\vec{a}\cdot\vec{c} = |a|\; P_{\vec{a}}(\vec{c}) = |a|\;|PQ|$

Just verify $\quad a\cdot(\vec{b}+\vec{c}) = |a|\;|OQ| = \underbrace{|a|\;|OP|}_{a\cdot b} + \underbrace{|a|\,|PQ|}_{a\cdot c}$

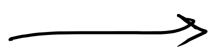
$a\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$

4) $\quad \vec{a}\cdot\vec{a} = |a|^2 \geq 0$

$\vec{a}\cdot\vec{a} = 0$ iff $\vec{a} = 0$

$\vec{a}\cdot\vec{a} = |\vec{a}|\,|\vec{a}|\,\cos\theta_{a,a}$

$= |\vec{a}|\,|\vec{a}|\,\cos 0 = |a|^2$


a

$\vec{a}\cdot\vec{a} = |a|^2 \geq 0$

& $\quad a\cdot a = 0$ if and only if $\vec{a} = 0$

Summary of properties of inner product:

1) $\quad \vec{a}\cdot(\lambda\vec{b}) = (\lambda\vec{a})\cdot\vec{b} = \lambda(\vec{a}\cdot\vec{b})$

1) $\vec{a} \cdot (\lambda \vec{b}) = (\lambda \vec{a}) \cdot \vec{b} = \lambda (\vec{a} \cdot \vec{b})$

scalar product homogeneity

2) $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

distributive w.r.t vector addition

3) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ Commutative

4) $\vec{a} \cdot \vec{a} \geq 0$ & $\vec{a} \cdot \vec{a} = 0 \iff \vec{a} = 0$

positive (definite)

These (vector space and inner product) will be the basis of their general definition that is NOT directly linked to any REAL VECTORs.

Coordinate system for vectors:

Linear independence:

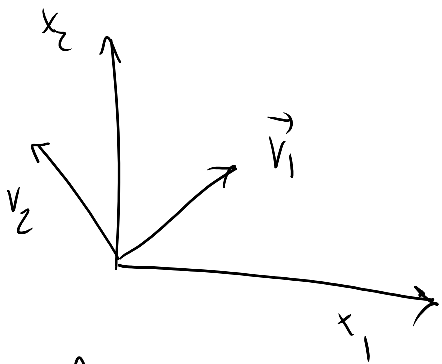
$\vec{V}_1, \vec{V}_2 \dots, \vec{V}_n$ are called linearly independent if

$$\lambda_1 \vec{v}_1 + \lambda_2 \vec{v}_2 + \dots + \lambda_n \vec{v}_n = 0 \implies \lambda_1 = \lambda_2 = \dots = \lambda_n = 0$$

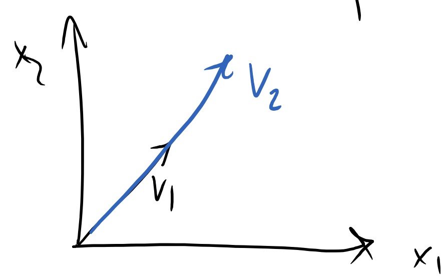
$$\left(\lambda_i \vec{v}_i = 0 \implies \lambda_i = 0\right)$$ indicial notation

$$\lambda_1 \vec{v}_1 + \lambda_2 \vec{v}_2 = 0 \implies$$

$$\lambda_1 = \lambda_2 = 0$$



v_1 & v_2 are linearly independent



$$2 V_1 + (-1) V_2 = 0$$

not linearly independent

Basis for a vector space:

$\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ is called a basis for a vector space if

1) $\vec{e}_1, \dots, \vec{e}_n$ are linearly independent

2) ANY vector can be written

a linear combination of basis vectors

as linear combination of basis vectors

$$\vec{V} = V_1 \vec{e}_1 + V_2 \vec{e}_2 + V_3 \vec{e}_3 + \dots + V_n \vec{e}_n$$

general vector

V_i coefficients are called

Coordinates of $\vec{V}$ w.r.t. $\{e\}$ basis

the number of terms in a basis is called the dimension of space

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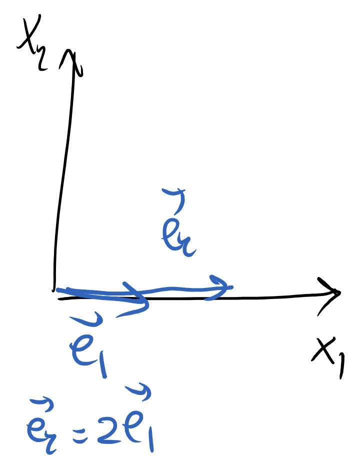
e_1, e_2 is not a basis

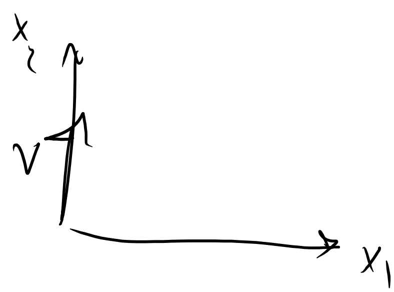
1) $e_2 = 2\vec{e}_1$

NOT linearly independent



2)

$$V = V_1 e_1 + V_2 e_2$$



impossible

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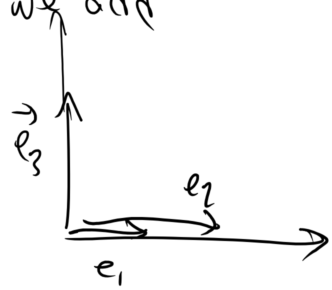
even if we add



$\{e_1, e_2, e_3\}$ is still not a basis because they are not linearly independent

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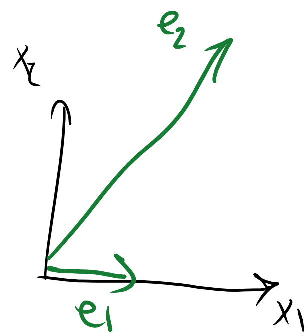
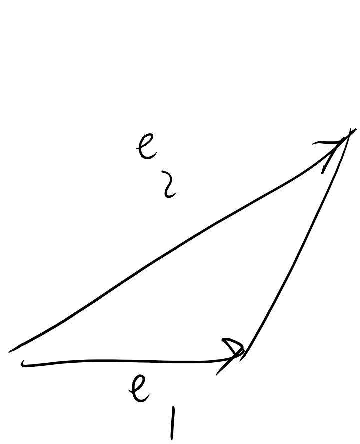
This is a valid basis



We often don't use this
basis because basis vectors are

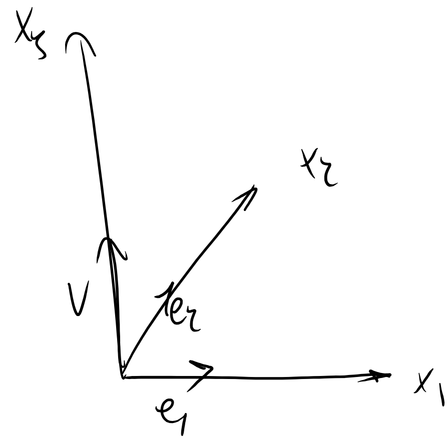
1. They are not normal
2. They are not unit size

$\{e_1, e_2$ is a good basis for this triangular geometry



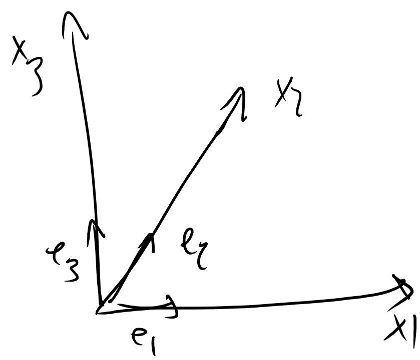
e_1, e_2 a basis No

does not cover $\vec{v}$



good basis

$$e_i \cdot e_j = \delta_{ij}$$



Orthonormal basis