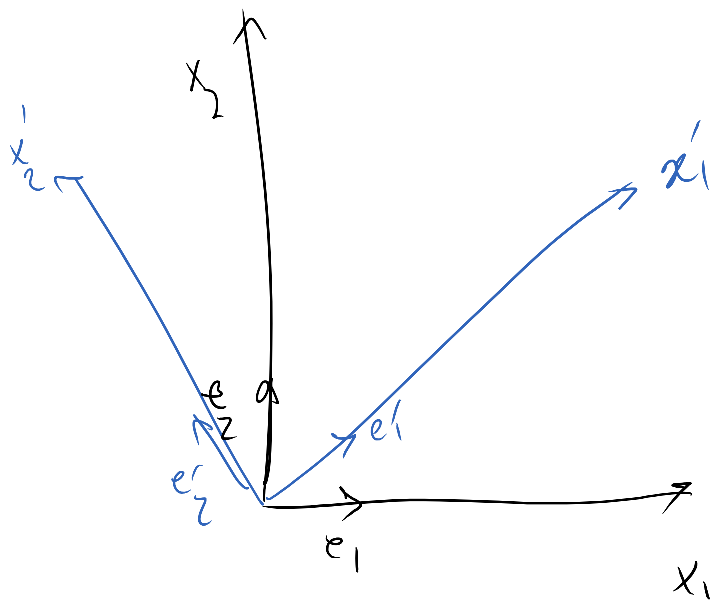
2017/09/07

Thursday, September 07, 2017 8:40 AM

Overview from last time:



$$Q = \begin{bmatrix} e'_1 \\ e'_2 \end{bmatrix}$$

expressed in e coordinate

$$= \begin{bmatrix} e_1 & e_2 \end{bmatrix}$$

expressed in e' coordinate

$\{e_1, e_2\}$
$\{e'_1, e'_2\}$ orthonormal basis

$$\left.\begin{aligned} e'_i &= Q_{ij} e_j \\ e_j &= Q_{ij} e_i \end{aligned}\right\} \rightarrow \left\{\begin{aligned} v'_i &= Q_{ij} v_j \\ v_j &= Q_{ij} v_i \end{aligned}\right.$$

Coordinate transformation
: vector components

A definition of a vector can be an n-tuple (size = dimension) that follows coordinate transformation rule.

Generalization of some of these ideas:

1. Vector Space

Vector space is a commutative Group with a scalar product operation.

Group:

A group is a set, G, together with an operation • (called the *group law* of G) that combines any two elements a and b to form another element, denoted $a \cdot b$ or ab. To qualify as a group, the set and operation, $(G, \cdot)$, must satisfy four requirements known as the *group axioms*:[5]

> The axioms for a group are short and natural... Yet somehow hidden behind these axioms is the monster simple group, a huge and extraordinary mathematical object, which appears to rely on numerous bizarre coincidences to exist. The axioms for groups give no obvious hint that anything like this exists.
>
> Richard Borcherds in *Mathematicians: An Outer View of the Inner World* [4]

Closure

For all a, b in G, the result of the operation, $a \cdot b$, is also in G.[b]

Associativity

For all a, b and c in G, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

Identity element

There exists an element e in G such that, for every element a in G, the equation $e \cdot a = a \cdot e = a$ holds. Such an element is unique (see below), and thus one speaks of *the* identity element.

Inverse element

For each a in G, there exists an element b in G, commonly denoted a^{-1} (or $-a$, if the operation is denoted "+"), such that $a \cdot b = b \cdot a = e$, where e is the identity element.

The result of an operation may depend on the order of the operands. In other words, the result of combining element a with element b need not yield the same result as combining element b with element a; the equation

$a \cdot b = b \cdot a$

What is a vector space?

$u, v, w \in$ vector space

Axiom	Meaning	
Associativity of addition	$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$	A1
Commutativity of addition	$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	A2
Identity element of addition	There exists an element $\mathbf{0} \in V$, called the *zero vector*, such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$	A3

Group

Associativity of addition	$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$	
Commutativity of addition	$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	A2
Identity element of addition	There exists an element $\mathbf{0} \in V$, called the *zero vector*, such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in V$.	A3
✩ Inverse elements of addition × not needed	For every $\mathbf{v} \in V$, there exists an element $-\mathbf{v} \in V$, called the *additive inverse* of $\mathbf{v}$, such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.	
Compatibility of scalar multiplication with field multiplication	$a(b\mathbf{v}) = (ab)\mathbf{v}$ [nb 2]	
Identity element of scalar multiplication	$1\mathbf{v} = \mathbf{v}$, where 1 denotes the multiplicative identity in F.	①
Distributivity of scalar multiplication with respect to vector addition	$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$	
Distributivity of scalar multiplication with respect to field addition	$(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$	90c

Group properties

additional parts

scalar product.

Examples of vector space:

Vectors in 1D, 2D, 3D (where the name comes from)

why ✩ is not needed

$$-V := \underbrace{(-1)}_{\text{scalar number } 1} V$$

$$V + (-V) = 0$$

inverse member for addition

How to prove this

property ①

$$V + -V = 0V + (-1)V = 1V + (-1)V$$

$$v + -v = \bigcirc v + (-1)v = (1)v + (-1)v$$

want to
create a scalar product

$$1v + (-1)v = (1+(-1))v = 0 \cdot v = 0$$

using OOC

?

$$0 \cdot v + v = 0 \cdot v + 1 \cdot v = (0+1) \cdot v = 1 \cdot v = v$$

$$\left.\begin{aligned} 0 \cdot v + v &= v \\ 0 + v &= v \end{aligned}\right\} \Longrightarrow \underline{0 \cdot v = 0}$$

In TAM551.pdf there are many proof examples related to vector spaces

Definition of vector spaces in TAM551 (pages 23, 24) - same definitions mentioned above for vector space

1. the set of real numbers $\Re$,
2. the standard algebraic operations of addition and multiplication defined over $\Re$ (each of these maps two real numbers into a real number),
3. a set $\mathcal{G}$ of elements called vectors (not necessarily vectors in Euclidean space!), that contains a zero vector, denoted $\mathbf{0}$,
4. an operation called vector addition, denoted $\overrightarrow{+}$, that maps two vectors into a vector,
5. an operation called scalar multiplication, denoted $*$, that maps a real number and a vector into a vector,

such that the following relations hold:

1. $\mathbf{a} \overrightarrow{+} \mathbf{b} = \mathbf{b} \overrightarrow{+} \mathbf{a}$ $\forall$ ("for all") $\mathbf{a}, \mathbf{b} \in \mathcal{G}$. (vector addition is commutative)
2. $\left(\mathbf{a} \overrightarrow{+} \mathbf{b}\right) \overrightarrow{+} \mathbf{c} = \mathbf{a} \overrightarrow{+} \left(\mathbf{b} \overrightarrow{+} \mathbf{c}\right)$ $\forall\, \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathcal{G}$. (vector addition is associative)
3. $\mathbf{a} \overrightarrow{+} \mathbf{0} = \mathbf{a}$ $\forall\, \mathbf{a} \in \mathcal{G}$. (property of the zero vector)

4. $(\lambda\mu) * \mathbf{a} = \lambda * (\mu * \mathbf{a})$ $\forall\, \mathbf{a} \in \mathcal{G}$ and $\forall\, \lambda, \mu \in \Re$. (scalar multiplication is associative)
5. $\lambda * \left(\mathbf{a} \overrightarrow{+} \mathbf{b}\right) = (\lambda * \mathbf{a}) \overrightarrow{+} (\lambda * \mathbf{b})$ $\forall\, \mathbf{a}, \mathbf{b} \in \mathcal{G}$ and $\forall\, \lambda \in \Re$. (scalar multiplication is distributive w.r.t. vector addition)
6. $(\lambda + \mu) * \mathbf{a} = (\lambda * \mathbf{a}) \overrightarrow{+} (\mu * \mathbf{a})$ $\forall\, \mathbf{a} \in \mathcal{G}$ and $\forall\, \lambda, \mu \in \Re$. (scalar multiplication is distributive w.r.t. addition of real numbers)
7. $\cancel{0 * \mathbf{a} = \mathbf{0}}$; $1 * \mathbf{a} = \mathbf{a}$ $\forall\, \mathbf{a} \in \mathcal{G}$.

Additional note (FYI):
The scalars come from a field
Field is mathematical structure where a set has addition and multiplication properties listed at:

[Field]

Real numbers and complex numbers form a field. We often use

REAL vector spaces where the scalars are real numbers, but we can also have complex vector fields.

Inner product:

Inner product is a function that takes two (general) vectors (a member of a vector space) and has the following properties:

1. $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$ Commutative

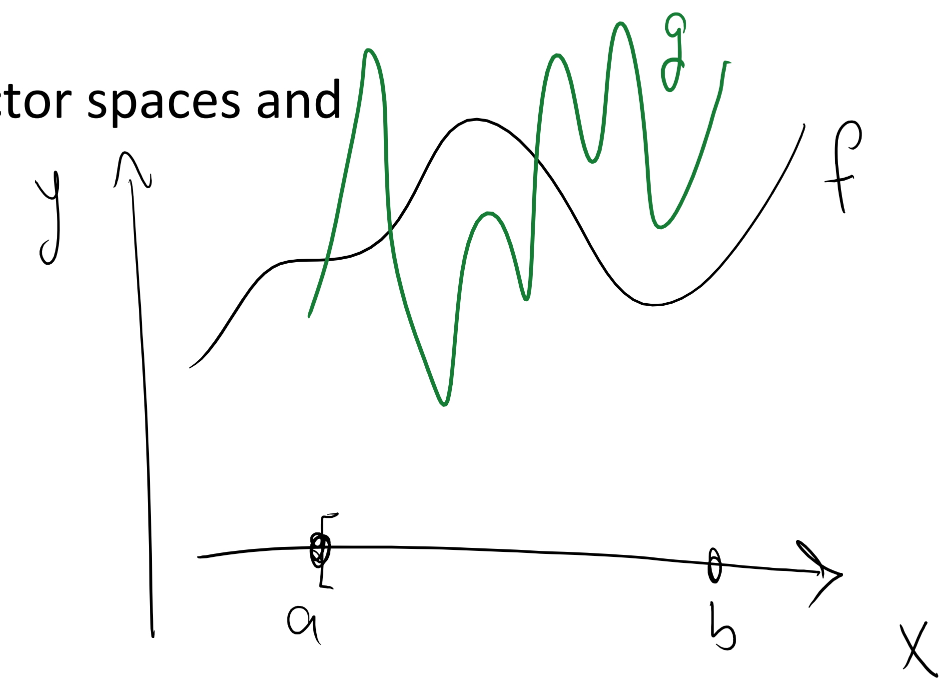
2. $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$ distributive

3. $\underbrace{(\lambda\vec{a})}_{\text{vector}}\cdot\underbrace{\vec{b}}_{\text{vector}} = \underbrace{\lambda}_{\text{scalar}}\underbrace{(\vec{a}\cdot\vec{b})}_{\text{scalar}}$ Inner product homogeneity

4. $\vec{a} \cdot \vec{a} \geq 0$ & $\vec{a} \cdot \vec{a} = 0$ iff $\vec{a} = 0$

We covered the same properties before for vectors in 2D and 3D and actually we proved 1 to 4 there. But inner product is more general and can be defined for metric vector spaces (i.e. vector spaces that have inner product)

Examples of other vector spaces and inner products:



A function is called L^2 integrable over $[a, b)$

if

$$\left| \int_a^b f^2 \, dx \right| < \infty$$

These functions form a vector space:

Most fns

$f+g \in V$

Vectorspace $\quad f, g, h \in L^2([a,b])$

1) $f+g = g+f$

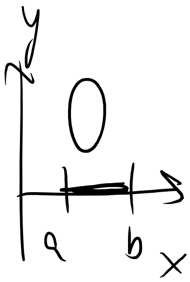
$(f+g)(x) := f(x)+g(x)$

2) $f+(g+h) = (f+g)+h$

3) $f+0 = 0+f = f$

where 0 function is

$0(x) = 0$
$x \in [a,b]$



4) $\lambda(\mu f) = (\lambda\mu) f$

5) $\lambda(f+g) = \lambda f + \lambda g$

6) $(\lambda+\mu) f = \lambda f + \mu f$

7) $1 \cdot f = f$

(λf):

$\forall x \in [a,b]$

$(\lambda f)(x) := \lambda f(x)$

lets prove this

Why $f+(g+h)=(f+g)+h$?

$\forall x \in [a,b]$

$\circledast$ Def. $(F+G)$:

$(F+G)(x) = F(x)+G(x)$

$\big(\underbrace{f}_{F} + \underbrace{(g+h)}_{G}\big)(x) =$

$f(x) + (g+h)(x) =$

$f(x) + \big(\underset{F}{g(x)} + \underset{G}{h(x)}\big) =$

Associative property of real numbers

$\big(f(x)+g(x)\big) + h(x) =$

by Def $\circledast$ $(f+g)(x) + h(x) =$

" " $\big((f+g)+h\big)(x)$

So $\forall x$ $\quad (f+(g+h))(x) =$

$$((f+g)+h)(x)$$

So $\quad f+(g+h) = (f+g)+h$

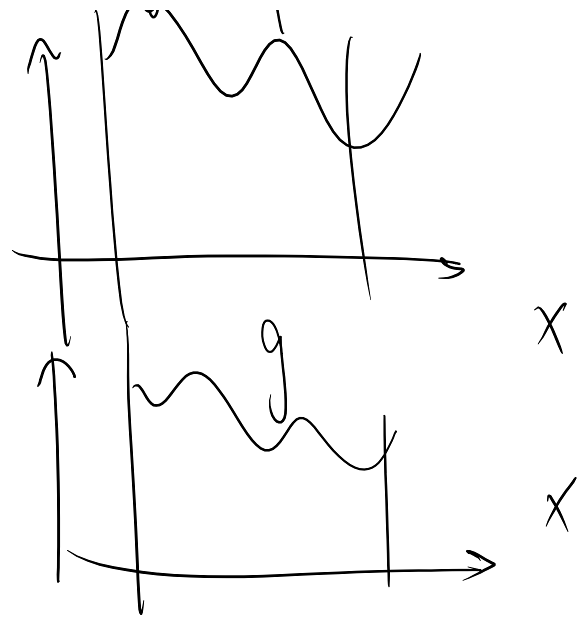
V: our vector space

More difficult question

Why if $f \in L^2([a,b])$ & $g \in L^2([a,b])$

$\underline{(f+g)} \in L^2([a,b])$

V = L2([a, b]) in fact is a metric (inner product) vector space, meaning that it has an inner product defined for it.



f

$f \cdot g$ = real number

$f \cdot g := \int_a^b f(x)\, g(x)\, dx$

defined

$\underbrace{\qquad\qquad}_{\text{is it a real number}}$

Yes

Does it have properties of inner product

1) $f \cdot g = g \cdot f$

2) $f \cdot (g+h) = f \cdot g + f \cdot h$

3) $(\lambda f) \cdot g = \lambda (f \cdot g)$

3) $(\lambda f) \cdot g = \lambda (f \cdot g)$

4) $f \cdot f \geq 0$ & $f \cdot f = 0$ iff $f = 0$

$$(\lambda f) \cdot g = \int_a^b \Big(\underbrace{(\lambda f)}_{\text{by def}} (x) \Big) g(x)\, dx$$

$$= \int_a^b \big(\lambda f(x)\big)\, g(x)\, dx$$

Real #s
$(ab)c = a(bc)$.
associative

$$= \int_a^b \lambda \big(f(x)\, g(x) \big)\, dx$$

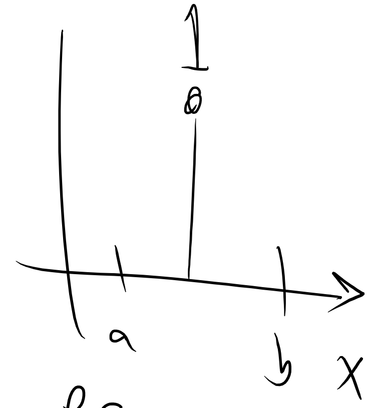
$$= \lambda \int_a^b f(x)\, g(x)\, dx$$

$\lambda \; f \cdot g$ $\boxtimes$

$(\lambda f) \cdot g = \lambda (f \cdot g)$

You can prove the other 3 properties

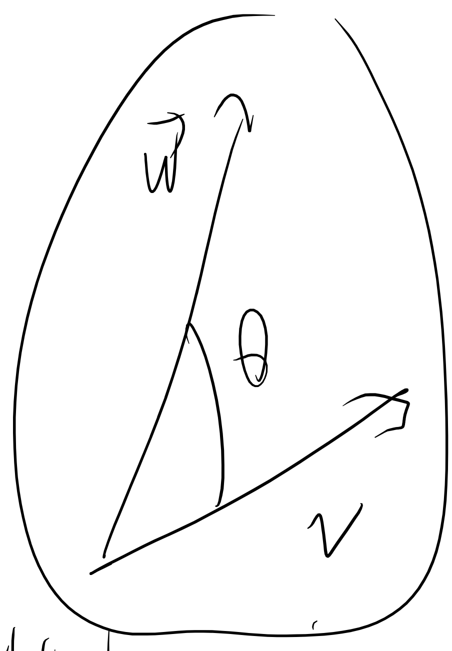
You can prove the other 3 properties similarly!
Hint: for property 4, assume function is also continuous



$f \cdot f = 0$ but $f \neq 0$
for non cont. function

Properties of General inner products

If v and w are vectors in 2D or 3D we had:

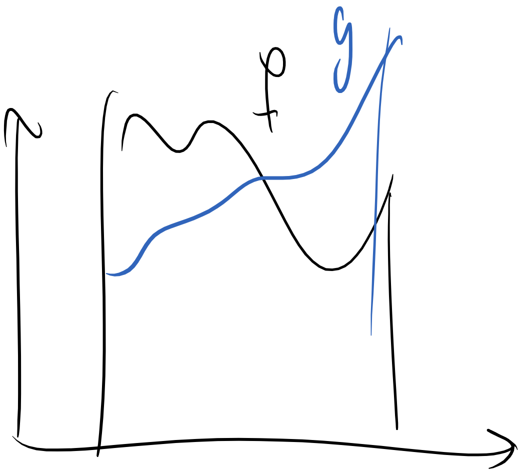


$$\vec{v} \cdot \vec{w} = |v||w|\cos\theta$$

$$|\vec{v} \cdot \vec{w}| = |v||w||\cos\theta| \le |v||w| \quad \text{Obvious}$$

Can we prove this identity in general?

Can we define magnitude in general?
Can we define angle between two vectors in general?



example

$$\Theta_{f,g} = 30^\circ \ !$$

$$\vec{v} \cdot \vec{w} = |v||w|\cos\theta$$

any vector

can we define angle

Easy part:
Definition of magnitude:

V is a general vector space with inner product $\circ$.

$$|a| := \sqrt{a \circ a}$$

Prop. 4 $\quad a \cdot a \geq 0$

$|a| = 0 \quad \text{iff} \quad a = 0$

Inner product inequality:

real numbers

magnitudes of f & f

In general

$$\underbrace{|f \cdot g|}_{\text{real number}} \leq |f|\,|g|$$

$$(\overbrace{f + \alpha g}^{h}) \cdot (\overbrace{f + \alpha g}^{h}) =$$

α real number

$$f \cdot f + \alpha g \cdot f + \alpha^2 g \cdot g + f \cdot \alpha g$$

$$= \underbrace{|f|^2}_{C} + \underbrace{(2 f \cdot g)}_{B}\alpha + \alpha^2 \underbrace{|g|^2}_{A} \geq 0 \quad h \cdot h \geq 0$$

$$A\alpha^2 + B\alpha + C \geq 0 \quad \text{for all } \alpha$$

$$\Delta = B^2 - 4AC < 0$$

$$(2\, f\cdot g)^2 - 4\,|g|^2 |f|^2 < 0$$

$$(f\cdot g)^2 < |f|^2 |g|^2 \implies$$

$$|f\cdot g| \le |f|\,|g| \quad \blacksquare$$

Now that we know this, we can define angle between any two members of an inner product vector space

$$\boxed{\cos \theta_{f,g} := \frac{f\cdot g}{|f|\,|g|}} \quad (\star)$$

Why does this makes sense?

$$|f \cdot g| \leq |f|\,|g| \Rightarrow \frac{|f \cdot g|}{|f|\,|g|} =$$

$$|\cos\theta| \leq 1$$

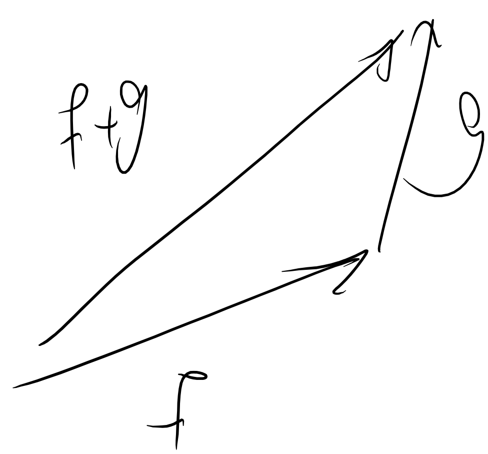
We used θ in actual vectors but for arbitrary vectors (e.g. functions) we still can define angle.

HW: one is in HW1 that you need to return?

$$|f+g| \leq |f| + |g|$$ triangular inequality

to prove it compute

$$(f+g)\cdot(f+g) \quad \& \quad \text{use } |f \cdot g| \leq |f|\,|g|$$



Why if f and g are square integrable so is f + g

$$\left.\begin{array}{l}\int_a^b |f|^2 < \infty \\ \int_a^b |g|^2 < \infty\end{array}\right\} \Longrightarrow |f+g| = \sqrt{\int_a^b (f+g)^2} < \infty$$

why ?

Hint: use inequality above

$$f, g \in L^2([a,b]) \Longrightarrow f+g \in L^2([a,b])\ ?$$

The answer is previous question

Norm

V is a vector space, and norm $\| v \|$ has the following properties:

$\forall\ v, w \in V \quad \lambda \in \mathbb{R}$

$$\begin{cases} 1) & \|\lambda v\| = |\lambda|\,\|v\| \\ 2) & \|v\| \geq 0 \\ 3) & \|v+w\| \leq \|v\| + \|w\| \quad \text{Triangular inequality} \end{cases}$$

If $\| \ \|$ exists & properties 1, 2, 3 hold

V is called a normed vector space

Q: What is we have an inner product? Is it a norm vector space?

We defined $|a| = \sqrt{a \cdot a}$

1) $|\lambda a| = |\lambda| |a|$

2) $|a| = \sqrt{a \cdot a} \geq 0$

$|a| = 0$ iff $a = 0$

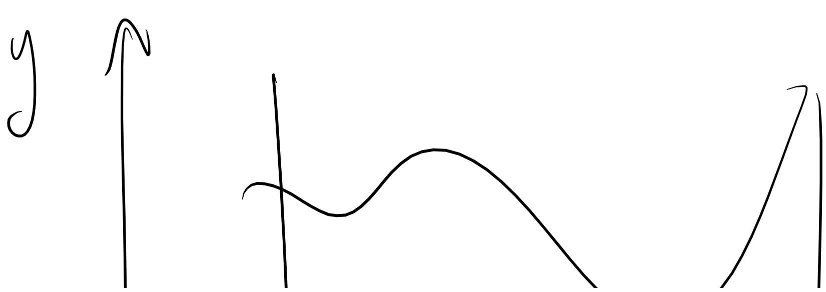
3) $|a+b| \leq |a| + |b|$

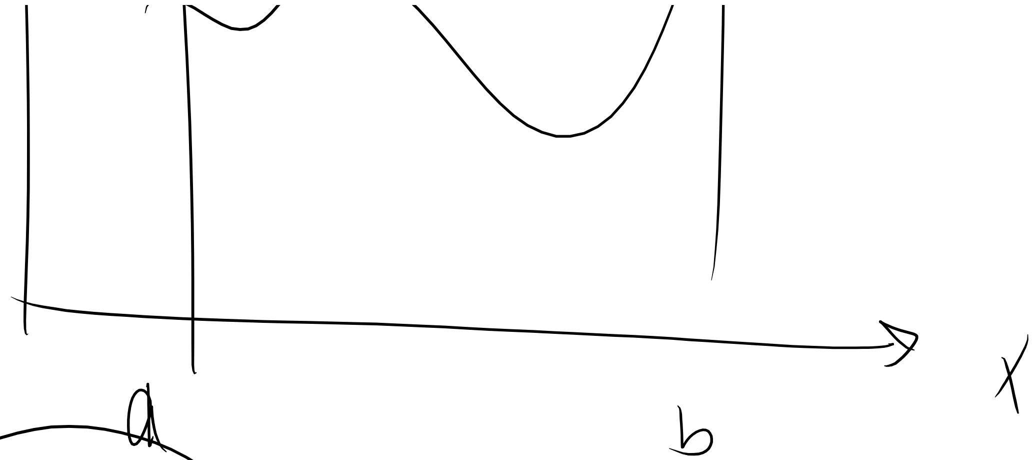
proved for inner product

An inner product space is a norm space (where norm is basically magnitude implied by inner product)

BUT THE INVERSE DOES NOT HOLD.

You can have a norm space that is not an inner product space. Basically we cannot define inner product out of a norm (whereas we could define a norm out of an inner product)





$$\|f\|_\infty = \max_{x\in[a,b]} \left(|f(x)| \right)$$

This is a norm but no inner product can be defined

$$L^p \quad \|f\|_p := \sqrt[p]{\int_a^b |f(x)|^p \, dx} < \infty$$

Functions with $L^p = \{ f \mid \|f\|_p < \infty \}$

Is a norm space but not an inner product space.