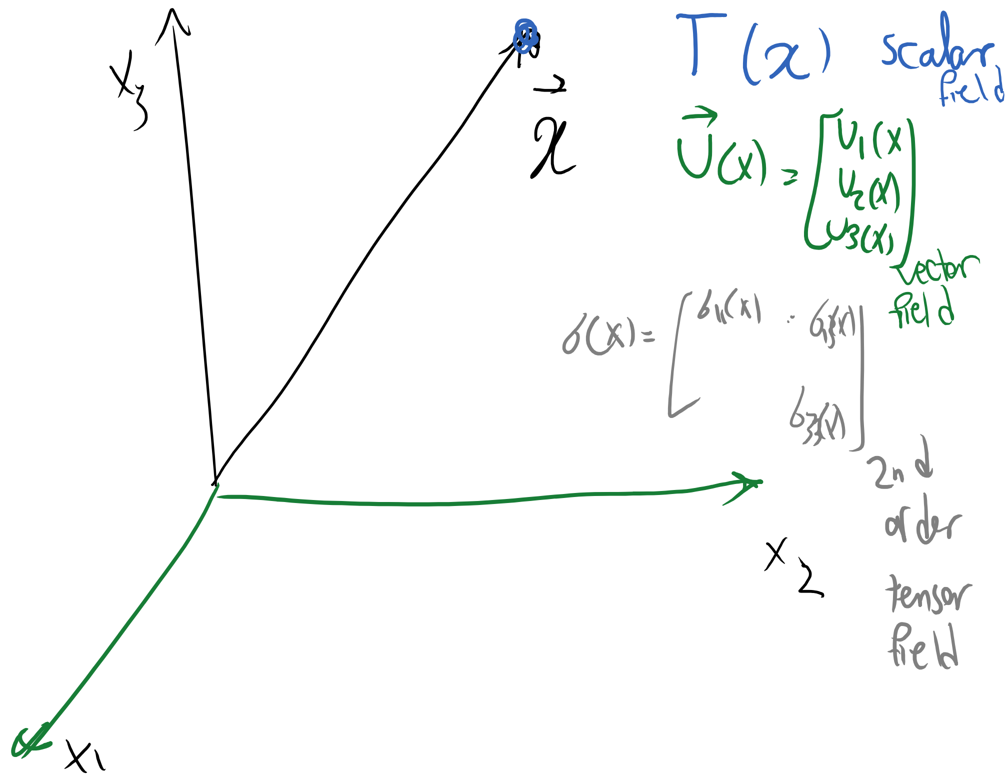
2017/09/26

Tuesday, September 26, 2017 8:38 AM

Tensor fields:



A tensor field is a function from Cartesian system to tensors. The tensors can be scalar, vector, second order tensor, or even higher.

Definition 57 *Let $\mathcal{D}$ denote a subset of a three-dimensional Euclidean point space $\mathcal{E}$ (we write $\mathcal{D} \subset \mathcal{E}$), and let* P *be an arbitrary point in $\mathcal{D}$. A function f that maps $\mathcal{D}$ to a set of tensors of order m is called an m^{th}*-order tensor field, *where $\mathcal{D}$ is called the* domain *of f. The value of f at* $\mathrm{P} \in \mathcal{D}$ *is denoted* $f(\mathrm{P})$.

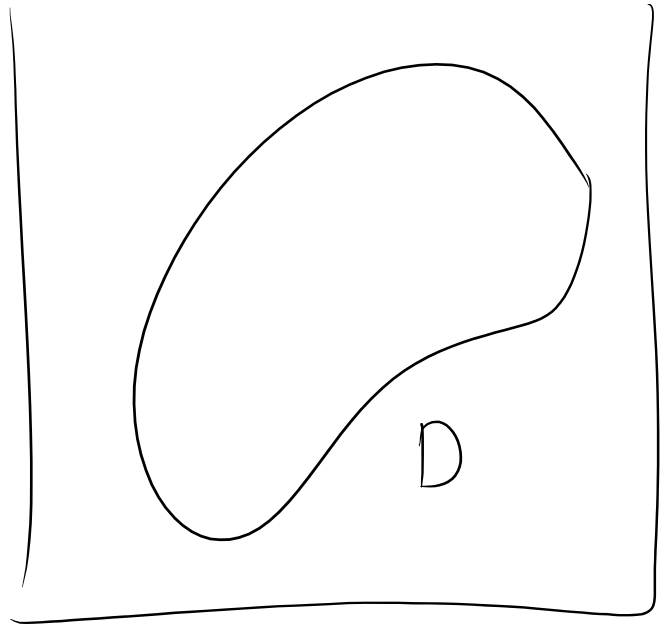
We use tensor fields in balance laws, kinematic equations, etc.

kinematic equations, etc.

$$\text{Mass} = \int_D \rho(\vec{x})\, dV$$

mass density

mass density is a scalar tensor field



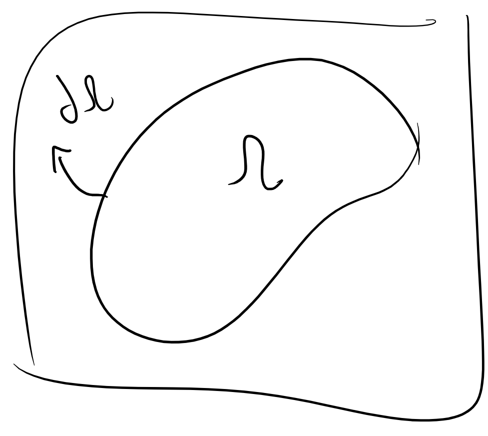
Many times we deal with derivatives of tensor fields

Balance of linear momentum

$$\int_\Omega \rho b\, dV + \int_{\partial\Omega} \vec{t}\, dS$$

traction vector

boundary of Ω



D

$\bar{T}\; n$

$$\vec{t} = \bar{\bar{\sigma}}\, n$$

stress tensor

divergence (Gauss)

$$\int_{\partial\Omega} \vec{t}\, dS = \int_{\partial\Omega} \sigma n\, dS = \int_{\Omega} \nabla\cdot\sigma\, dV$$

so we need to compute derivative of tensor fields

Continuity notation:

$T(\vec{x})$ tensor field is $C^m(D,\mathbb{R})$

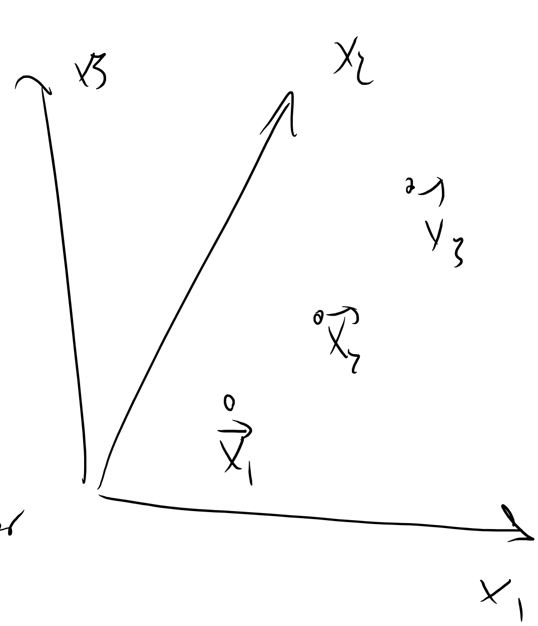
T is scalar

$C^m(D,\mathbb{R}^3)$

T is a vector

$C^m(D,\mathbb{R}^3\times\mathbb{R}^3)$

T is 2nd order tensor, etc



C^m Means that each component of the

$C^{3\,1}$ Means that each component of the tensor has n continuous derivatives w.r.t. all coordinate components

$\vec{v}$ is vector $\quad v \in C^{2}(D, R^{3})$

$V_{1,11} \quad V_{1,12} \quad V_{1,13} \quad V_{1,21} \quad V_{1,22} \quad V_{1,23} \quad V_{1,31} \quad V_{1,32} \quad V_{1,33}$

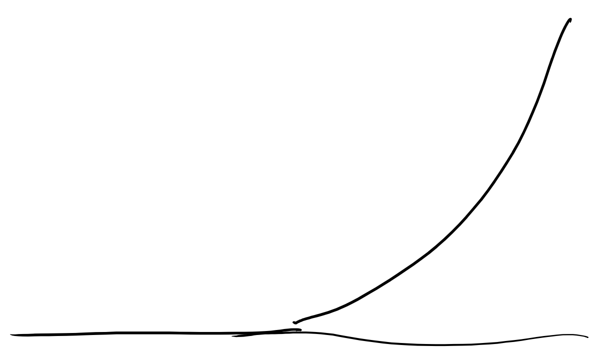
$V_{3,11} \qquad\qquad\qquad\qquad V_{3,33}$

these values exist and are all continuous

In 1D $\quad f(x) = \begin{cases} 0 & x < 0 \\ x^2 & x \geq 0 \end{cases}$

$\underline{C^{1} \text{ function}}$

$f(x)$

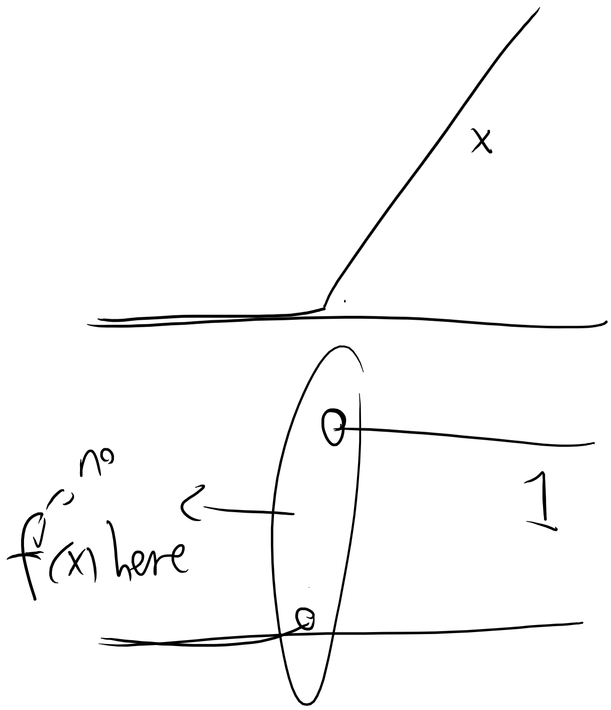


C^{c} ✓

$f'(x)$

C^1

$f''(x)$



$\cancel{C^2}$

Definition of partial derivatives:

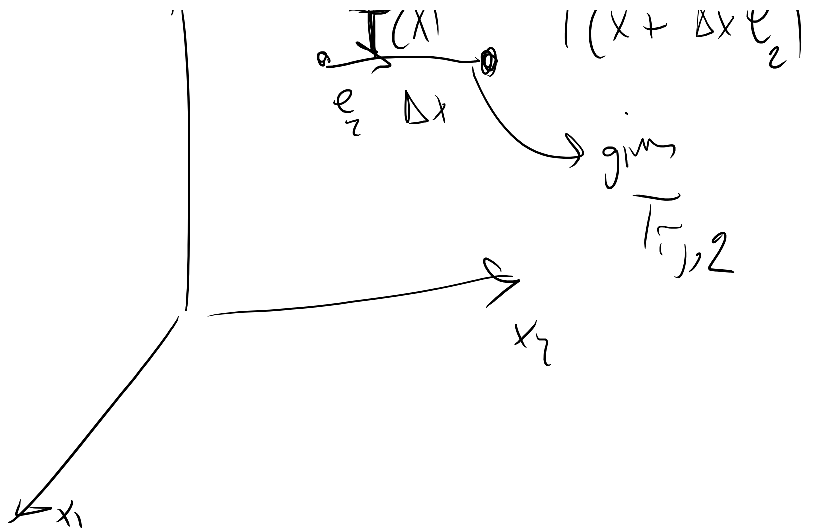
T 2nd order tensor

$$T = T_{ij}\, e_i \otimes e_j$$

$$T_{ij,k} = \frac{\partial T_{ij}(x)}{\partial x_k} = \lim_{\Delta x \to 0} \frac{T_{ij}(x + e_k \Delta x) - T(x)}{\Delta x}$$

the new thing

the new thing



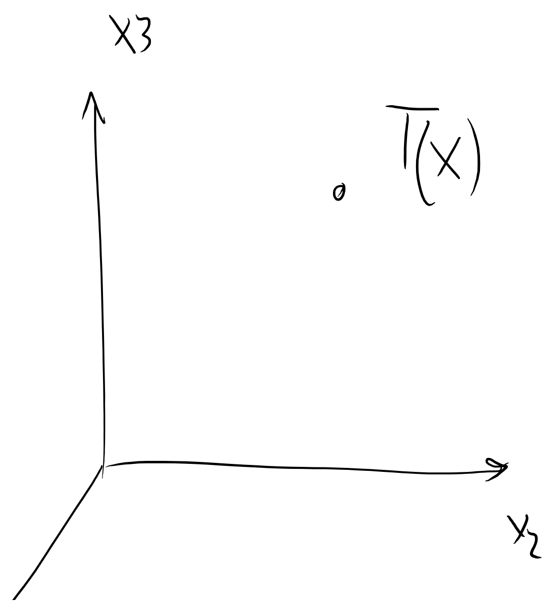
This is a generalization of partial derivatives for scalars which we are familiar with

$$\phi_{,k} = \lim_{\Delta x \to 0} \frac{\phi(x + e_k \Delta x) - \phi(x)}{\Delta x}$$

Let's say we have a Cartesian coordinate system, we want to define gradient of a tensor

$$T = T_{i_1 i_2 \cdots i_n} e_{i_1} \otimes \cdots \otimes e_{i_n}$$

$$\underbrace{\nabla T} = T_{i_1 i_2 \cdots i_n, j} \, e_{i_1} \otimes \cdots \otimes e_{i_n} \otimes e_j$$



gradient of T

ϕ scalar $\qquad \nabla\phi = \phi_{,1}\, e_1 + \phi_{,2}\, e_2 + \phi_{,3}\, e_3$

$$= [\phi_{,1} \quad \phi_{,2} \quad \phi_{,3}]$$

covector

u is a vector

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \qquad \nabla u = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} \\ u_{2,1} & u_{2,2} & u_{2,3} \\ u_{3,1} & u_{3,2} & u_{3,3} \end{bmatrix}$$

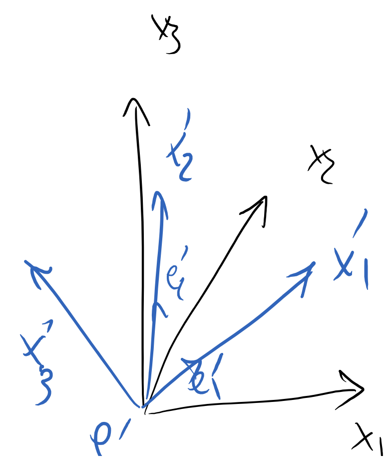
How do we know if T is a tensor, it's gradient is also a tensor?

I'll show this for a second order tensor

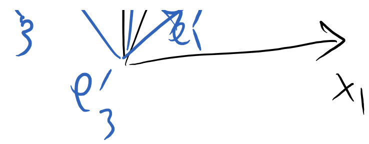
$$T = T_{ij}\, e_i \otimes e_j$$

$$(\nabla T)_{ijk} = T_{ij,k} = \frac{\partial T_{ij}(x_1, x_2, x_3)}{\partial x_k}$$

$$(\nabla T)' = T'_{mn} \ldots = \partial T'_{mn}(x'_1, x'_2, x'_3)$$



$$(\nabla T)'_{mnp} = T'_{mn,p} = \frac{\partial T'_{mn}(x'_1, x'_2, x'_3)}{\partial x'_p}$$



To show grad is a tensor, we should show

$$(\nabla T)'_{mnp} = Q_{mi}\, Q_{nj}\, Q_{pk}\, (\nabla T)_{ijk}$$

$$T'_{mn} = Q_{mi}\, Q_{nj}\, T_{ij}$$

because T is a tensor

$$(\nabla T)'_{mnp} = \frac{\partial T'_{mn}}{\partial x'_p} = \frac{\partial \left(\overbrace{Q_{mi}\, Q_{nj}}^{\text{constant}}\, T_{ij}(x) \right)}{\partial x'_p}$$

$$= Q_{mi}\, Q_{nj}\, \frac{\partial T_{ij}(x_1, x_2, x_3)}{\partial x'_p}$$

using chain rule

$$= Q_{mi} Q_{nj} \underbrace{\left\{ \frac{\partial \tilde{T}_{ij}(x)}{\partial x_k} \cdot \underbrace{\frac{\partial x_k}{\partial x'_p}}_{Q_{pk}} \right\}}_{\text{summation over } k \text{ (repeated } k)} \qquad \text{I}$$

$$x_k = Q_{sk}\, x'_s$$

$$\frac{\partial x_k}{\partial x'_p} = \cancel{\frac{\partial Q_{pk}\, x'_p}{\partial x'_p}}$$

3 times repeated indices

$$\frac{\partial x_k}{\partial x'_p} = \frac{\partial Q_{sk}\, x'_s}{\partial x'_p} = Q_{sk} \frac{\partial x'_s}{\partial x'_p}$$

$$= Q_{sk}\, \delta_{sp} = Q_{pk}$$

From (I) $\Longrightarrow$

$$(\nabla T)'_{mnp} = Q_{mi} Q_{nj} Q_{pk} \frac{\partial T_{ij}}{\partial x_k}$$

$$(\nabla T)'_{mnp} = Q_{mi} Q_{nj} Q_{pk} (\nabla T)_{ijk}$$

So T is a tensor so it its gradient

Consequences:

1. Anything that follows transformation rule is frame-independent (objective), that is it has physical interpretation independent of the coordinate system used to express it.
2. If we have partial derivative w.r.t. one coordinate system, we have partial derivative in any other coordinate system.

That's why if we show for example ui,j

exist and are continuous for ONE coordinate system, they exist in any other coordinate system

$$u'_{m,n} = Q_{mi} Q_{nj} u_{i,j}$$

That's why we can define C^m continuity conditions mentioned before

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What are some of the uses of gradient, divergence, curl, …

$$\underbrace{\nabla \cdot T}_{\text{divergence}}$$

$$v, \quad \nabla v = v_{i,j}\, e_i \otimes e_j$$

$$\underbrace{\nabla \cdot v} = v_{i,i}$$

Always contract the last two indices

$$T = T_{ij}\, e_i \otimes e_j \qquad \nabla T = T_{ij,k}\, e_i \otimes e_j \otimes e_k$$

Contract

$$\operatorname{div} T = \nabla \cdot T$$

$\mathrm{div}\, l = \nabla \cdot l$

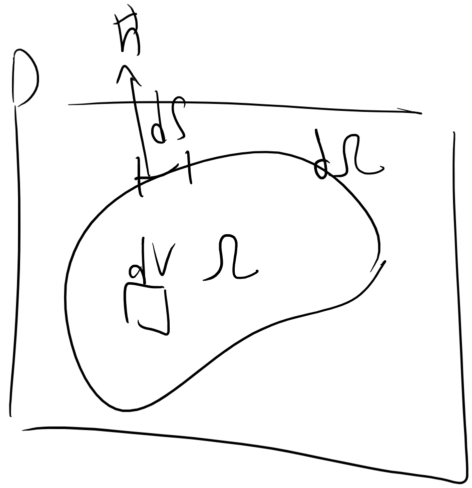
Contract j&k

$$\boxed{\nabla \cdot T = T_{i,jj}\, e_i}$$

$\mathrm{Curl}\, v = \nabla \times v$

$$\nabla = \frac{\partial}{\partial y_1} e_1 + \frac{\partial}{\partial x_2} e_2 + \frac{\partial}{\partial x_3} e_3$$

Some useful theorems:



1. Divergence theorem:

$$\oint_{\partial \Omega} T \cdot n \, dS = \int_{\Omega} (\nabla \cdot T)\, dV$$

Also called Gauss theorem

2. Curl theorem (or Stokes theorem - which in fact is more general)

$$\oint \vec{v} \cdot \vec{dt} = \int (\nabla \times v) \cdot \vec{n}\, ds$$

Curvilinear coordinate systems:

Curvilinear systems:

Please refer to (ref1) Appendix C of http://geo.mff.cuni.cz/vyuka/Martinec-ContinuumMechanics.pdf

(the text is in zip files sent to you in

Continuum References_Curvilinear\Martinec_Zdenek_Charles_U_Prague_Martinec-ContinuumMechanics.pdf) -

Refer to (ref2) http://homepages.engineering.auckland.ac.nz/~pkel015/SolidMechanicsBooks/Part_III/Chapter_1_Vectors_Tensors/Vectors_Tensors_14_Tensor_Calculus.pdf

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(shared with you in the same same zip files)

for a discussion on the meaning of grad operator.

For your information (beyond the class) This is an excellent reference on curvilinear coordinates in general: http://www.mech.utah.edu/~brannon/public/curvilinear.pdf

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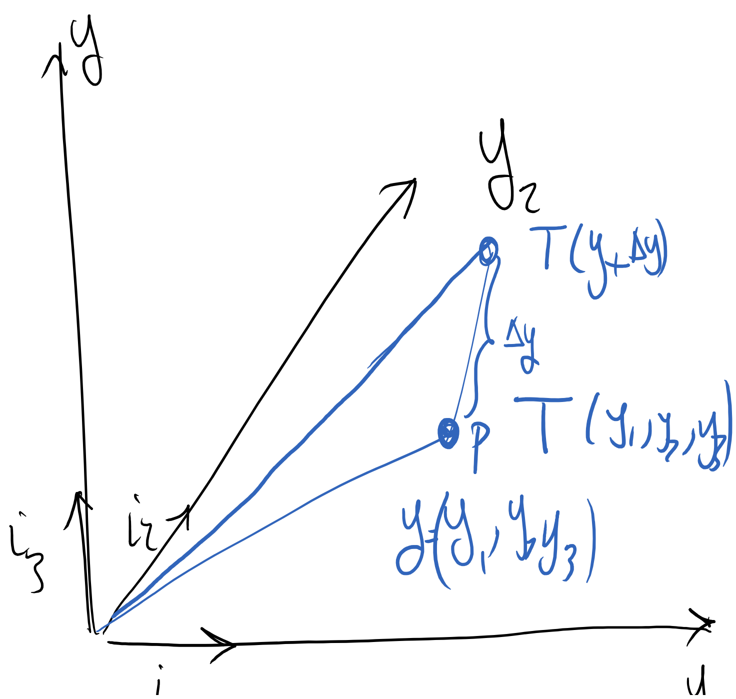
Notations:

y_1, y_2, y_3

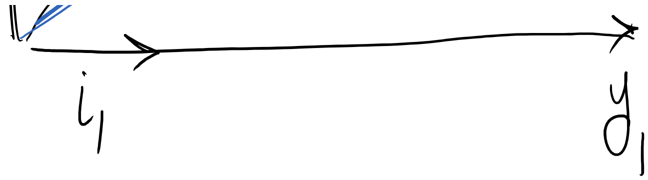
are Global Cartesian

coordinate values

i_1, i_2, i_3



i_1, i_2, i_3
the corresponding unit vectors



change in position $\Delta P = \Delta y$

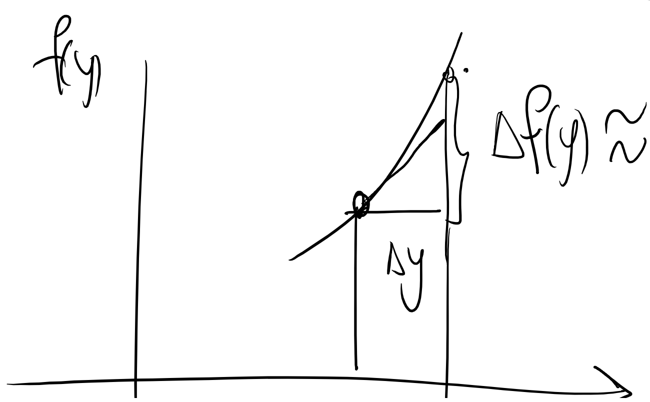
We are given the change in coordinate Δy

We want to obtain the change in tensor value $\Delta T = ?$

(Δ : means change, "increment". It's not Laplacian operator)

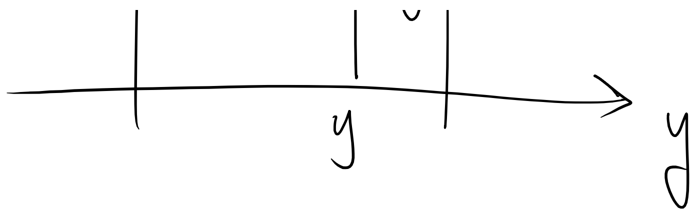
$$\underbrace{\Delta T}_{\text{change in } T} \approx \left(\quad \right) \underbrace{\Delta P}_{\text{change in position}}$$

1D



$$\Delta f(y) \approx f'(y)\, \Delta y$$

derivative



Derivative maps change in position to change in function value.

Gradient does the same thing, but in d= 1, 2, 3 and for any tensor order

T: n th order tensor

$$\underbrace{\Delta T}_{n\text{th order tensor}} \approx \text{grad}\, T \ \underbrace{\Delta P}_{\text{vector}}$$

Must be $\underline{n+1}$ order tensor

$$n = (n+1) - 1$$

$$T(y+\Delta y) = T(y_1+\Delta y_1,\ y_2+\Delta y_2,\ y_3+\Delta y_3)$$

$$= T(y_1, y_2, y_3) + \frac{\partial T}{\partial y_1}\Delta y_1 + \frac{\partial T}{\partial y_2}\Delta y_2 + \frac{\partial T}{\partial y_3}\Delta y_3$$

$$= \underbrace{T(y_1, y_2, y_3)}_{T(y)} + \underbrace{\frac{\partial T}{\partial y_1}\Delta y_1 + \frac{\partial T}{\partial y_2}\Delta y_2 + \frac{\partial T}{\partial y_3}\Delta y_3}_{\text{1st increment in Taylor expansion}}$$

$$\underbrace{+ \frac{1}{2!}\frac{\partial T}{\partial y_1 \partial y_1}\Delta y_1^2 + \frac{\partial T}{\partial y_1 \partial y_2}\Delta y_1 \Delta y_2 + \ldots + \frac{1}{2}\frac{\partial T}{\partial y_3 \partial y_3}\Delta y_3^2}_{\text{2nd increment}}$$

$$+ O(\Delta y^3)$$

discuss this notation later

$$\Delta T = T(y + \Delta y) - T(y) = \begin{bmatrix} \frac{\partial T}{\partial y_1} & \frac{\partial T}{\partial y_2} & \frac{\partial T}{\partial y_3} \end{bmatrix} \begin{bmatrix} \Delta y_1 \\ \Delta y_2 \\ \Delta y_3 \end{bmatrix}$$

gradient

$$\frac{1}{2}\begin{bmatrix} \Delta y_1 & \Delta y_2 & \Delta y_3 \end{bmatrix} \underbrace{\begin{bmatrix} T_{,11} & T_{,12} & T_{,13} \\ T_{,21} & T_{,22} & T_{,23} \\ T_{,31} & T_{,32} & T_{,33} \end{bmatrix}}_{\text{Hessian}} \begin{bmatrix} \Delta y_1 \\ \Delta y_2 \\ \Delta y_3 \end{bmatrix}$$

$$+ O(\Delta y^3)$$

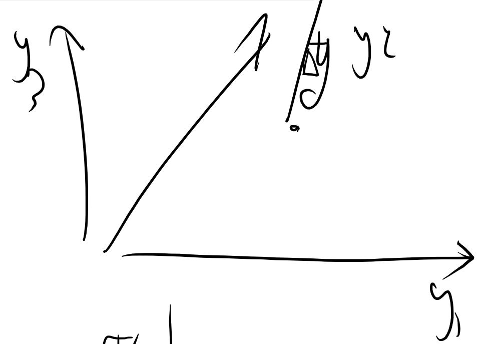
$$\boxed{\Delta T \approx \text{grad}\, T \, \Delta P}$$

change in position

$\Delta T \approx \text{grad}\, T \,(\quad)$ — change in position

$$\text{grad}\, T = \begin{bmatrix} \frac{\partial T}{\partial y_1} & \frac{\partial T}{\partial y_2} & \frac{\partial T}{\partial y_3} \end{bmatrix} \quad \text{covector}$$

$$\Delta P = \begin{bmatrix} \Delta y_1 \\ \Delta y_2 \\ \Delta y_3 \end{bmatrix}$$



Do it yourself:

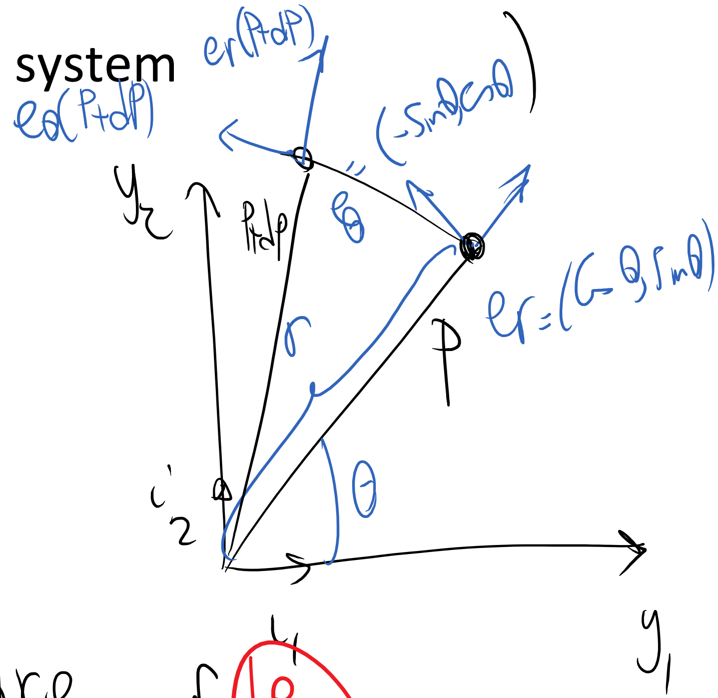
For a vector v = v1 i1 + v2 i2 + v3 i3 $\quad v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

$$\text{grad}\, v = \nabla v = \begin{bmatrix} v_{1,1} & v_{1,2} & v_{1,3} \\ v_{2,1} & v_{2,2} & v_{2,3} \\ v_{3,1} & v_{3,2} & v_{3,3} \end{bmatrix}$$

$$\Delta v = \begin{bmatrix} v_{1,1} & v_{1,2} & v_{1,3} \\ v_{2,1} & v_{2,2} & v_{2,3} \\ v_{3,1} & v_{3,2} & v_{3,3} \end{bmatrix} \begin{bmatrix} \Delta y_1 \\ \Delta y_2 \\ \Delta y_3 \end{bmatrix}$$

Grad in cylindrical coordinate system

in 2D



$$P = r\vec{e_r}$$

$$\underbrace{dP}_{\text{change in position}} = d(r\vec{e_r}) = dr\,e_r + r\,de_r$$

the unit vector changes by position

$$e_r = \cos\theta\, i_1 + \sin\theta\, i_2$$

$$de_r = -\sin\theta\, d\theta\, i_1 + \cos\theta\, d\theta\, i_2$$

$$= d\theta\,(\underbrace{-\sin\theta\, i_1 + \cos\theta\, i_2}_{e_\theta})$$

$$de_r = d\theta\, e_\theta$$

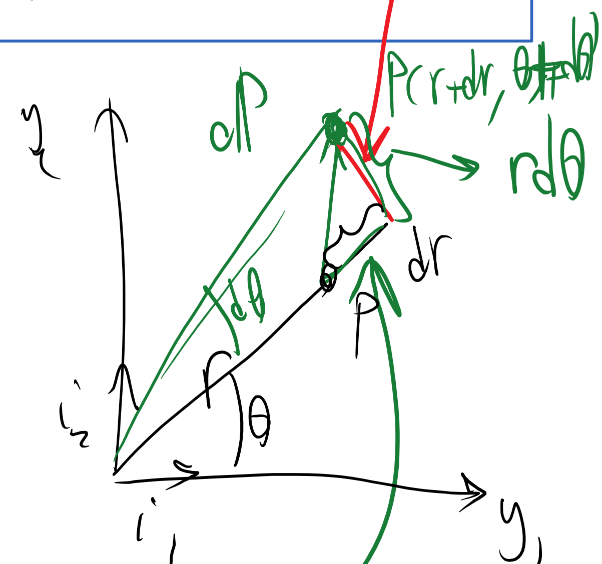
similarly $de_\theta = -d\theta\, e_r$

$de_r = d\theta\, e_\theta$ similarly $de_\theta = -d\theta\, e_r$

$dP = d(r e_r) = dr\, e_r + r\, de_r$

$\Rightarrow$ $\boxed{dP = dr\, e_r + (r\, d\theta)\, d\theta}$

$$dP = \begin{bmatrix} dr \\ r\, d\theta \end{bmatrix}$$ in e_r, e_θ coordinate system



Compare this with

$P = y_1 i_1 + y_2 i_2$

fixed

$dP = dy_1\, i_1 + dy_2\, i_2$

$\begin{bmatrix} dy_1 \end{bmatrix}$

$$= \begin{bmatrix} dy_1 \\ dy_2 \end{bmatrix}_{i, i_2 \text{ system}}$$