2019/09/04

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Indicial notation:

- If an index is not repeated it's going to be replaced by {1, 2, 3} or in general the dimension of the problem (2D {1, 2})
- If an index is repeated twice we do a summation on that
- Indices cannot be repeated more than 2 times in an expression (exception for eigen-expression)
- Indices should be consistent for all expressions -> if index i is given once in one expression in an equation it should be repeated once for all the other expressions and so forth

$a_i > 0$ means $\begin{cases} a_1 > 0 \\ a_2 > 0 \\ a_3 > 0 \end{cases}$

1 Repetition

$$C_{ii} = \sum_{i=1}^{3} C_{ii} = C_{11} + C_{22} + C_{33}$$ trace of C

we don't write this

$C_{11} > 0$

$C_{ii} > 0$ trace > 0

what about $\forall i \quad C_{ii} > 0$

$C_{i(i)} > 0 \qquad C_{i\underline{i}} > 0$

No summation applies here

$c_i + b_j = 5$ ✗

1i 0i

0j 1j

$c_i + c_{ii} = 10$

$\cancel{1}$ 2

$c_{ii} = a_i b_i \qquad \underline{a_i}$ dum ✓

$$\hat{\ } \quad c_{ij} = \underbrace{a_i b_j}_{} + \frac{a_i}{d_j} d_{kk} \quad \checkmark$$

$i\ 1 \qquad 1i \qquad 1i$

$j\ 1 \qquad 1j \qquad 1j$

$b = Aa \qquad c = ?\, a$

$c = Bb$

$c = (BA)a$

$$\left.\begin{array}{l} b_i = A_{ij} a_j \\ c_i = B_{ij} b_j \end{array}\right\} \Longrightarrow c_i = B_{ii}\, B_{ji}\, a_i \quad \times$$

$$\boxed{b_j = A_{ji} a_i}$$

$3\ i$'s

$b_j = A_{jk} a_k \Longrightarrow c_i = B_{ij} A_{jk} a_k$

$= (BA)_{ik} a_k$

$$\boxed{(BA)_{ik} = B_{ij} A_{jk}}$$

Make sure indices don't repeat more than twice

Eigen-expressions → ending up with 3 indices

A symmetric

$$\overset{1i}{A u_i} = \overset{2i}{\lambda_i u_i} \quad \text{no summation}$$

no summation

λ_i, u_i are eigen value, vector #i

↓

$$A = \begin{bmatrix} v_1 \\ \hline v_2 \\ \hline v_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$$

$$\boxed{A = \sum_{i=1}^{3} u_i^T \lambda_i u_i}$$

$3\ i$'s

we need to explicitly write the summation

$$\left. \right|_{i=1}$$

we need to explicitly write the summation

Kronecker delta:

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \qquad \delta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Identity matrix

1) $\delta_{ii} = \left(\sum_{i=1}^{3} \delta_{ii} \right) = 3$

2) $\delta_{\bar{i}\bar{i}} = 1$ no summation

3) $$\boxed{\delta_{ij} C_{mnpqjr} = C_{mnpqir}}$$

replaced by i

$$\delta_{ij} b_j = b_i$$

Permutation symbol

- Cross product
- Determinant

$$\epsilon_{ijk} \in \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$$

$i \in \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$ $j \in \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$

27 component

Only 6 are nonzero

Only 6 are nonzero

$\epsilon_{123} = 1$ $\quad \epsilon_{231} = 1$ $\quad \epsilon_{312} = 1$

$\epsilon_{321} = -1$ $\quad \epsilon_{213} = -1$ $\quad \epsilon_{132} = -1$

Anytime one of the indices is repeated we get 0

$\epsilon_{112} = 0 \quad \epsilon_{223} = 0, \ldots$

$$\det A = \det \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{array}{ll} \epsilon_{123} & + A_{11} A_{22} A_{33} \\ \epsilon_{231} & + A_{12} A_{23} A_{31} \\ \epsilon_{312} & + A_{13} A_{21} A_{32} \\ \epsilon_{132} & - A_{11} A_{23} A_{32} \\ \epsilon_{213} & - A_{12} A_{21} A_{33} \\ \epsilon_{321} & - A_{13} A_{22} A_{31} \end{array}$$

$$\boxed{\begin{aligned} \det A &= \epsilon_{ijk} A_{1i} A_{2j} A_{3k} \quad \leftarrow 1 \\ &= \epsilon_{ijk} A_{i1} A_{j2} A_{k3} \quad \leftarrow 2 \end{aligned}}$$

$$\epsilon_{mnp} \det A = \epsilon_{ijk} A_{im} A_{jn} {}_{kp}$$

$m=1$ $n=2$ $p=3$ $\longrightarrow$ get expression 2

$m=1$ $n=1$ $p=3$ LHS $\epsilon_{113} \det A = 0$

if indices are repeated $\Rightarrow$ $e = 0$ & det of such matrix is zero

(A) $\det \left[v_1 \middle| v_2 \middle| v_3 \right] = - \det \left[v_2 \middle| v_1 \middle| v_3 \right]$

and property (A) to show eqn (3) is correct of 5 other cases where m, n, p are not equal

What is juxtapose both rows and columns:

$$\epsilon_{ijk}\,\epsilon_{pqr}\,\det A = \det \begin{array}{c} i \\ j \\ k \end{array} \begin{bmatrix} a_{ip} & a_{iq} & a_{ir} \\ a_{jp} & a_{jq} & a_{jr} \\ a_{kp} & a_{kq} & a_{kr} \end{bmatrix}$$

$A = \delta$

$$\boxed{\epsilon_{ijk}\,\epsilon_{pqr}\,\underbrace{\det \delta}_{1} = \det \begin{bmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{bmatrix}}$$

$p = i$

Relates δ & ϵ

$p = i$

$$\boxed{\epsilon_{ijk}\,\epsilon_{iqr} = \delta_{jq}\,\delta_{kr} - \delta_{jr}\,\delta_{kq}}$$

$q = j$

$$\epsilon_{ijk}\,\epsilon_{ijr} = \delta_{jj}\,\delta_{kr} - \delta_{jr}\,\delta_{kj}$$
$$= 3\delta_{kr} - \delta_{kr}$$
$$= 2\delta_{kr}$$

$$\boxed{\epsilon_{ijk}\,\epsilon_{ijr} = 2\,\delta_{kr}}$$

$k = r$

$$\boxed{\epsilon_{ijk}\,\epsilon_{ijk} = 2\underbrace{\delta_{kk}}_{3} = 6}$$

$\epsilon_{mnp}\,\times$

$$\epsilon_{mnp}\,\det A = \epsilon_{ijk}\det A\, A_{im}\,A_{jn}\,A_{kp}$$

$$\underbrace{\epsilon_{mnp}\,\epsilon_{mnp}}_{6}\,\det A = \epsilon_{mnp}\,\epsilon_{ijk}\,A_{im}\,A_{jn}\,A_{kp}$$

$$\det A = \frac{1}{6}\,\epsilon_{mnp}\,\epsilon_{ijk}\,A_{im}\,A_{jn}\,A_{kp}$$

$$\boxed{\det A = \frac{1}{6} \epsilon_{mnp} \epsilon_{ijk} A_{im} A_{jn} A_{kp}}$$

$$\boxed{A^{-1}_{ip} = \frac{1}{2 \det A} \epsilon_{ijk} \epsilon_{pqr} A_{qr} A_{rk}}$$

HW: $A^{-1}_{ip} A_{ps} = \delta_{is}$ verify this

$Q = x^T A x$ — A matrix, x vector

Scalar $x_i (A_{ij} x_j)$

i & j are dummy indices, repeated twice

$(Ax)_i = A_{ij} x_j$

$x_i (Ax)_i = x_i A_{ij} x_j$

$Q(x_1, x_2, x_3)$ $\qquad \frac{\partial Q}{\partial x_j}$

$$\frac{\partial Q}{\partial x_j} = \frac{\partial (A_{ij} x_i \cancel{x_j})}{\partial x_j} = A_{ij} x_i$$

j is repeated 3 times

Correct way

$$k \quad \frac{\partial Q}{\partial x_k} = \frac{\partial A_{ij} x_i x_j}{\partial x_k} = A_{ij} \left(\underbrace{\frac{\partial x_i}{\partial x_k}}_{\delta_{ik}} \right) x_j + x_i \underbrace{\frac{\partial x_j}{\partial x_k}}_{\delta_{jk}}$$

$$= \underbrace{A_{ij} (\delta_{ik}}\, x_j + x_i \delta_{jk})$$

$$\underbrace{A_{ij} x_i \delta_{jk}}$$

$$= A_{kj} x_j + A_{ik} x_i$$

$j \quad i$

$$= A_{ki} x_i + A_{ik} x_i = (A_{ki} + A_{ik}) x_i$$

$$= A_{kj} x_j + A_{jk} x_j = (A_{kj} + A_{jk}) x_j$$

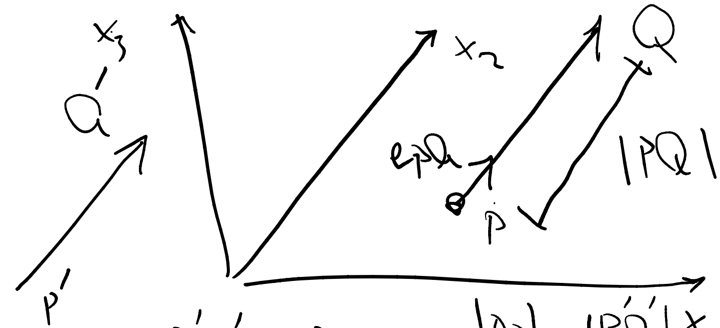
Vector spaces:

$\vec{PQ}$

1 – length $\quad |PQ|$

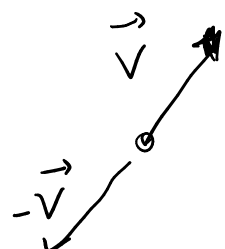
2 – direction $\quad e_{PQ}$

3 – absolute position $\quad P$

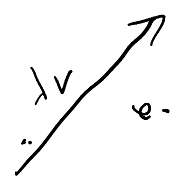


$P'Q' = PQ \qquad |PQ| = |P'Q'| \qquad e_{PQ} = e_{P'Q'}$

In most applications we don't care about the base position of the vector and only use 1 and 2 to check for equality of two vectors



$a - b = a + (-b)$



$\lambda \in \mathbb{R} \quad \lambda > 0 \quad$ e.g. $\lambda = 2$

Properties

Addition

$a + b = b + a$ commutative

$a + (b + c) = (a + b) + c$ associative

$a + 0 = 0 + a = a$ zero vector

Scalar product $\lambda \geq 0$

$\lambda = -.5$

Scalar product properties

$(\lambda\mu) a = \lambda(\mu a)$ scalar product associative

$(\lambda + \mu) a = \lambda a + \mu a$ distributive w.r.t scalar addition

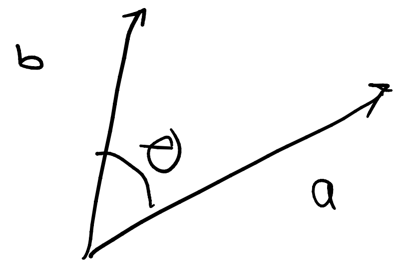
$\lambda(a + b) = \lambda a + \lambda b$ " " vector addition

$1 \cdot a = a$

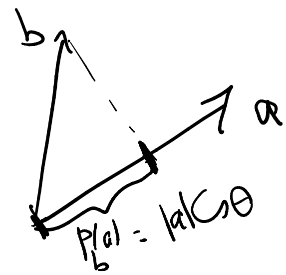
$a + (-a) = 0$ we can prove this

Inner product between two vectors:

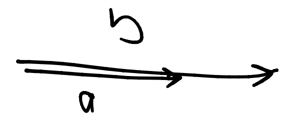
$$a \cdot b = |a||b| \cos\theta$$
$$= (P_b a)\,|b|$$



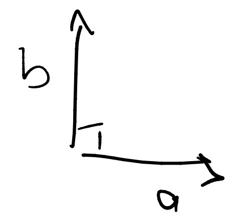
(a, b)
$\langle a, b \rangle$ Inner product notations



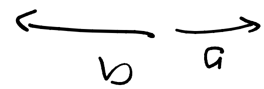
Maximum value of $a \cdot b = |a||b|$ $\theta = 0$



Normal vectors $a \cdot b = 0$



Minimum value $a \cdot b = -|a||b|$ $\theta = 180$



Properties of inner product

1. $\vec{a} \cdot (\lambda \vec{b}) = \lambda \vec{a} \cdot \vec{b} = \lambda (a \cdot b)$ Scalar product homogeneity
2. $a \cdot (b + c) = a \cdot b + a \cdot c$ distributive w.r.t vector addition
3. $a \cdot b = b \cdot a$ Commutative
4. $a \cdot a \geq 0$ & $a \cdot a = 0 \iff a = 0$ positive definite properties