If a vector space has an inner product operation, we call it an inner product vector space

\[ \langle f, g \rangle = \int_a^b f(x)g(x) \, dx \]

This is the L2 space (square integrable functions) in \([a, b]\) defined as

\[ \mathcal{L}^2([a, b]) = \{ f : \int_a^b |f(x)|^2 \, dx < \infty \} \]

We can show that \( \mathcal{L}^2([a, b]) \) is a vector space.

\[ \langle f, g \rangle = \int_a^b f(x)g(x) \, dx < \infty \]

\[ \langle f, f \rangle = \int_a^b f(x)^2 \, dx = \| f \|_2^2 < \infty \]

\[ \langle g, g \rangle = \int_a^b g(x)^2 \, dx < \infty \]

This is the L2 space (square integrable functions) in \([a, b]\) defined as

\[ \mathcal{L}^2([a, b]) = \{ f : \int_a^b |f(x)|^2 \, dx < \infty \} \]

This is a subspace (a subset of a vector space that itself is a vector space) of \( \mathcal{V} \) and in fact it is an inner product vector space.

\[ \| f \|_2 = \left( \int_a^b |f(x)|^2 \, dx \right)^{1/2} \]
Remember for vectors \( \langle v, w \rangle = |v||w| \cos \theta \)

\[ \cos \theta = \frac{\langle v, w \rangle}{|v||w|} \]

From here we can define the angle between \( f, g \) as

\[ \cos \theta_{fg} = \frac{\langle f, g \rangle}{\sqrt{|f|^2} \sqrt{|g|^2}} \]

We know \( |\cos \theta| \leq 1 \). How do we know \( |\langle f, g \rangle| \leq |f||g| \)?

Cauchy-Schwarz Inequality

**Proof:**

Given any inner product vector space \( \mathbb{V} \) over \( \mathbb{F} = \mathbb{R} \) or \( \mathbb{C} \),

\[ \langle f + \alpha g, f + \alpha g \rangle \geq 0 \quad (1) \]

\[ \langle f + \alpha g, f + \alpha g \rangle = \langle f, f \rangle + 2\Re(\langle f, \alpha g \rangle) + |\alpha|^2 \langle g, g \rangle \quad (3) \]

\[ f, f + \alpha(g, f) + \alpha (g, f) + |\alpha|^2 g, g \]

\[ = |f|^2 + |\alpha|^2 |g|^2 + 2\Re(f, g) \]

\[ = |f|^2 + |\alpha|^2 |g|^2 + 2\Re(f, g) \]

\[ \geq 0 \quad (4) \]

This is a quadratic polynomial in \( \alpha \),

\[ A\alpha^2 + B\alpha + C \geq 0 \quad \Delta = B^2 - 4AC < 0 \]
\[(2f \cdot g)^2 - 4(g \cdot g) (f \cdot f) \leq 0\]

\[0 \leq (f \cdot g)^2 \leq (f \cdot f) (g \cdot g)\] take square root

\[|f \cdot g| \leq |f| \cdot |g|\] 

Triangle inequality: \(H^n, \ell^1\)

\[\forall f, g \quad |f + g| \leq |f| + |g|\]

\[\text{Hint: work with the square of } \otimes \text{ & use (CS) after that.}\]

Going back to \(L^2(a,b)\)

Why \(\langle f, f \rangle = \int_a^b f^2 \, dx < \infty\) \(\quad \Rightarrow \int_a^b g \, dx < \infty\)

\[\langle f, g \rangle \leq \sqrt{\langle f, f \rangle} \sqrt{\langle g, g \rangle} < \infty\]

Normed vector space:

From an inner product we defined "magnitude" as:

\[|f| = \sqrt{\langle f, f \rangle}\]

1. \(|f| = |\|f\|_1|\)

2. \(|f| \geq 0\) \& \(f = 0 \Rightarrow |f| = 0\)

'absolute value' operator on vector
- A norm operator has exactly the 3 properties listed above.
- Unlike inner product that acts on two vectors, norm acts only on ONE vector
- Any inner product defined a norm (its magnitude operator is a norm) BUT the opposite is not true (Cannot always define an inner product (space) from a norm (space)).

General definition of a **norm**:
For a vector space V, a norm has the following properties

\[ \forall v \in V, \quad \|v\| = |\lambda| \|v\| \]

2) \( \|v\| \geq 0 \) & \( \|v\| = 0 \) if\( v = 0 \)

3) \( \|v + w\| \leq \|v\| + \|w\| \) **triangle inequality**

An example of a normed space that is not an inner product space

\[
L^\infty([a,b]) = \left\{ f \mid \max_{x \in [a,b]} |f(x)| < \infty \right\}
\]

\[
\|f\|_\infty = \max_{x \in [a,b]} |f(x)|
\]

\[
\|f\|_\infty = \text{max} \mid f \mid_{x \in [a,b]}
\]

\[
\|f\|_\infty \quad \text{this is a norm}
\]

but \( L^\infty([a,b]) \) is not an inner product space.

Coordinates and coordinate transformation:

Linear independence:

\[
V_1, V_2, \ldots, V_n \in V \quad \text{are called linearly independent if}
\]

\[
\forall \} \quad \text{vector space}
\]
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For \(\alpha_1, \ldots, \alpha_n \neq 0\):

\[ \alpha_1 v_1 + \ldots + \alpha_n v_n = 0 \quad \Rightarrow \quad \alpha_1, \ldots, \alpha_n \text{ must be zero.} \]

\[ 2v_1 - v_2 = 0 \]

Not independent.

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Basis \( e = \{e_1, \ldots, e_n\} \) for vector space \( V \) has the following properties:

1. \( e_1, \ldots, e_n \) are linearly independent.
2. Any vector \( v \) can be expressed in terms of the basis.

Dimension of vector space \( n \) is coordinate of \( v \) w.r.t. \( e \) coordinate system.

Examples for basis:

\[ v_1 \]

\[ v_2 \]

\( e_1, e_2 \) are not satisfied property 1 is violated.

\[ (5,5) \]

Natural coordinate for triangle (for all simplices).
Interpretation of coordinates of a vector in a given orthonormal basis

\[ V_i = \text{?} \quad \text{in an orthonormal basis} \]

\[ V = V_i \cdot e_i \]

\[ V \cdot e_j = (V_i \cdot e_i) \cdot e_j = \delta_{ij} \]

\[ = V_i \cdot (e_i \cdot e_j) \]

Orthonormality of the basis

\[ V_i = V \cdot e_i = |V| \cos \theta_i; \quad |e_i| = P_{e_i} \]

Coordinate transformation:

\[ V = V_1 e_1 + V_2 e_2 \]

\[ (V_1, V_2) \]

\[ V \]

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\[ \mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 \]

\[ \mathbf{v} = \mathbf{v}_1 \mathbf{e}_1 + \mathbf{v}_2 \mathbf{e}_2 \quad (\mathbf{v}_1, \mathbf{v}_2) \]

\[ (\mathbf{v}_1, \mathbf{v}_2) \rightarrow (\mathbf{v}_1', \mathbf{v}_2') = ? \]

In general, \((20,31)\) we can write

\[ e_i' = Q_{ij} e_j \]

\[ e_i' \cdot e_k = (Q_{ij} e_j) \cdot e_k = Q_{ij} e_j \cdot e_k = Q_{ij} S_{jk} \]

\[ e_i' = Q_{ij} e_j \quad \rightarrow \quad Q_{ij} = e_i' \cdot e_j \]

Component \(j\) of \(e_i'\) in coordinate system \(\{e_1, \ldots, e_l\}\)

\[ Q = \begin{bmatrix} e_1' & e_2' & e_3' \end{bmatrix} \]

\[ e_i' = Q_{ij} e_j \quad Q_{ij} = e_i' \cdot e_j \]

\[ \mathbf{R}_{ij} = \mathbf{Q}_{ii} \]
\[ e_i = Q_{ij} e_j \quad \Rightarrow \quad Q_{ij} = e_i \cdot e_j \]

\[ e_j = R_{ji} e_i \]

\[ \Rightarrow \quad R_{ji} = e_j \cdot e_i \]

\[ e_i' = Q_{ij} e_j \]

\[ e_j = Q_{ji} e_i \]

\[ \{ e_i' \} = Q \{ e_i \} \quad \Rightarrow \quad \{ e_i' \} = Q^{-1} \{ e_i \} \]

\[ \{ e_j \} = Q^T \{ e_j \} \]

\[ Q^{-1} = Q^T \]

\[ Q \text{ is an orthogonal matrix} \]

\[ Q^T Q = (Q^{-1} Q) Q^T = I \]

\[ Q^T = \begin{bmatrix} e_1' & e_2' & e_3' \\ e_2' & e_3' & e_1' \\ e_3' & e_1' & e_2' \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & e_3 \\ e_2 & e_3 & e_1 \\ e_3 & e_1 & e_2 \end{bmatrix} \]

\[ e_i' - e_j = \delta_{ij} \]

What is the use of \( Q \)?

\[ V = V_i e_i \]

\[ e_j = Q_{ij} e_i \]

\[ \Rightarrow \quad V = (Q_{ij}) e_i \]

\[ V = \begin{bmatrix} e_1' \\ e_2' \\ e_3' \end{bmatrix} \]
In general for $n^{th}$ order tensor

\[ T_{i_1 \ldots i_m} = Q_{i_1 j_1} Q_{i_2 j_2} \ldots Q_{i_m j_m} T_{j_1 \ldots j_m} \]