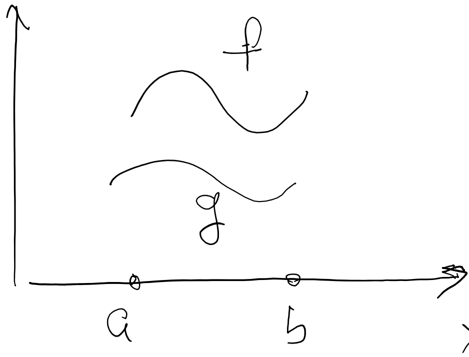
2019/09/09

Thursday, September 5, 2019 5:20 PM

If a vector space has an inner product operation, we call it an inner product vector space

$0 : \quad 0(x) = 0 \qquad x \in [a,b]$



$(\lambda f)(x) = \lambda (f(x))$

scalar # function

$(f+g)(x) = f(x) + g(x)$

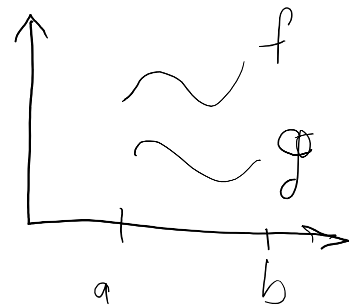
We can show $V = \{\text{function on } [a,b]\}$ is a vector space

$$\langle f,g \rangle = \int_a^b f \cdot g \, dx < \infty$$

(f,g)

$f \cdot g$

want to be able to define this integral & have a finite value



$$\langle f,f \rangle = \int_a^b f^2 dx = \|f\|^2 < \infty$$

$$\langle g,g \rangle = \int_a^b g^2 dx < \infty$$

$$\boxed{\|v\| = \sqrt{\langle v,v \rangle}}$$

any inner product defines a norm

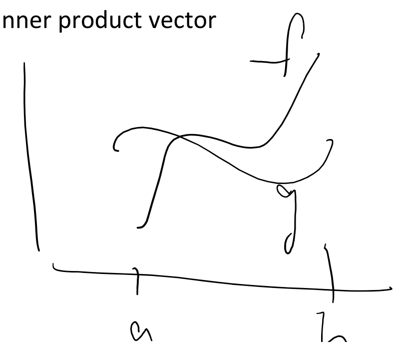
This is the L2 space (square integrable functions) in [a, b] defined as

$$W = \left\{ \text{functions } f \text{ in } [a,b] \;\middle|\; \int_a^b f^2 dx < \infty \right\} \quad L^2(a,b)$$

L_2

This is a subspace (a subset of a vector space that itself is a vector space) of V and in fact it is an inner product vector space.

$$\langle f,g \rangle = \int_a^b f \cdot g \, dx$$

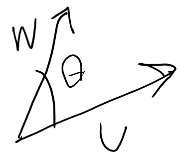


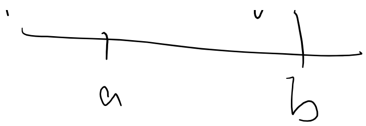
Remember for vectors we defined inner product as

$\langle v,w \rangle = \|v\| \|w\| \cos\theta$

Remember for vectors we

$\langle v, w \rangle = |v||w| \cos\theta$





$$\cos\theta = \frac{\langle v, w \rangle}{|v|\,|w|}$$

from here we can define angle between f, g

$$\boxed{\cos\theta_{f,g} = \frac{\langle f, g \rangle}{\underbrace{\sqrt{\langle f, f \rangle}}_{|f|}\sqrt{\langle g, g \rangle}}}$$

we know $|\cos\theta| \leq 1$. How do we know

$$\boxed{|\langle f, g \rangle| \leq |f|\,|g|}$$

Cauchy Schwarz inequality

Proof:

$f, g \in W$ (any inner product vector space)

$\alpha \in \mathbb{R}$

Properties of inner product $\forall \lambda \in \mathbb{R}$, $f, g, h \in W$
(1) $f \cdot g = g \cdot f$
(2) $(f+g) \cdot h = f \cdot g + f \cdot h$
(3) $(\lambda f) \cdot g = \lambda (f \cdot g)$
(4) $f \cdot f \geq 0$ $\quad f \cdot f = 0 \equiv f = 0$

$\underbrace{(f + \alpha g)}_{h} \cdot (f + \alpha g) \geq 0$ (4)

$(f + \alpha g) \cdot f + (f + \alpha g) \cdot \alpha g$ (2)

$f \cdot f + (\alpha g) \cdot f + f \cdot (\alpha g) + (\alpha g) \cdot \alpha g$ (2)

$f \cdot f + \alpha (g \cdot f) + \alpha f \cdot g + \alpha^2 g \cdot g =$

3 $\quad \underset{f \cdot g}{\overset{\|}{}}$ (1) $\quad$ 3 $\quad$ 3 2 times

$= \underbrace{\alpha^2 g \cdot g}_{A} + \alpha \underbrace{(2 f \cdot g)}_{B} + \underbrace{f \cdot f}_{C} \geq 0$ $\quad$ 2nd order polynomial in α

$\forall \alpha$

$A\alpha^2 + B\alpha + C \geq 0 \qquad \Delta = B^2 - 4AC \leq 0$

$$(2f\cdot g)^2 - 4(g\cdot g)(f\cdot f) \le 0$$

$$0 \le (f\cdot g)^2 \le (f\cdot f)(g\cdot g)$$ take square root

$$\boxed{|f\cdot g| \le |f|\,|g|} \quad (CS)$$

Triangle inequality: HW, FYI

$\forall f, g$ $\qquad |f+g| \le |f| + |g| \quad (\star)$

Hint work with the square of $(\star)$ & use (CS) after that.

going back to $L^2(a,b)$

why
$$\left.\begin{aligned} \langle f, f\rangle &= \int_a^b f^2\,dx < \infty \\ \langle g, g\rangle &= \int_a^b g^2\,dx < \infty \end{aligned}\right\} \Longrightarrow \int_a^b f\cdot g\,dx < \infty$$

$$\langle f, g\rangle \le \sqrt{\langle f, f\rangle}\,\sqrt{\langle g, g\rangle} < \infty$$

Normed vector space:

From an inner product we defined "magnitude" as:

$$|f| = \sqrt{\langle f, f\rangle}$$

① $|\lambda f| = \underbrace{|\lambda|}_{\text{absolute value operator}}\,|f|$

② $|f| \ge 0$

& $f=0$ iff $f=0$

'magnitude' operator on vector

③ $\forall f, g \quad |f+g| \leq |f| + |g|$ Triangle inequality

- A norm operator has exactly the 3 properties listed above.
- Unlike inner product that acts on two vectors, norm acts only on ONE vector
- Any inner product defined a norm (its magnitude operator is a norm) BUT the opposite is not true (Cannot always define an inner product (space) from a norm (space).

notation for norm

$$\underbrace{\|\cdot\|}$$

General definition of a **norm**:
For a vector space V, a norm has the following properties

$\forall v$

1) $\|\lambda v\| = |\lambda|\,\|v\|$

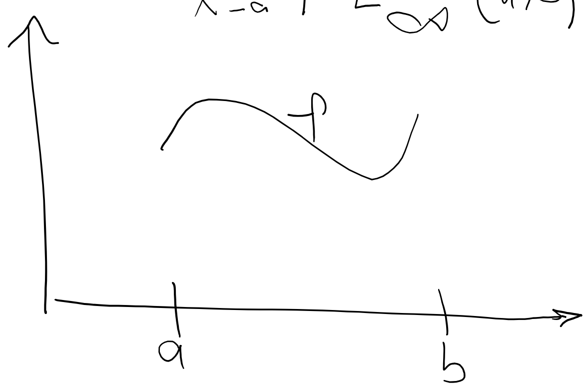
2) $\|v\| \geq 0$ & $\|v\| = 0$ iff $v = 0$

3) $\|v+w\| \leq \|v\| + \|w\|$ triangle inequality

$\frac{1}{x-a} \notin L_\infty(a,b)$

An example of a normed space that is not an inner product space

$$L_\infty(a,b) = W = \underset{\text{finite functions}}{\{f \mid Max(|f|) < \infty\}} = \{f \mid \|f\|_\infty < \infty\}$$



$$\underbrace{\|f\|_\infty} = \underset{x \in [a,b]}{Max} |f|$$

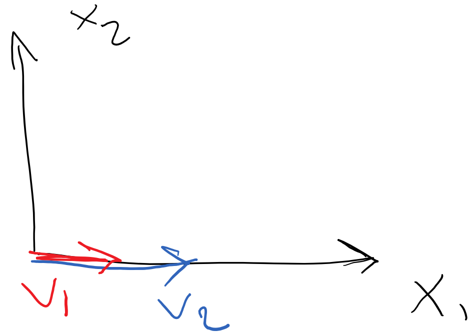
$\|\cdot\|_\infty$ this is a norm

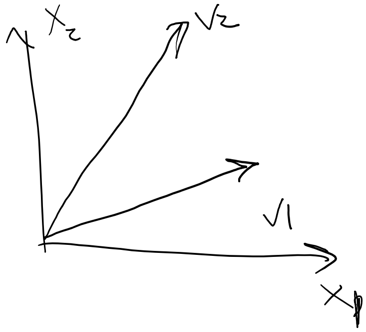
but $L_\infty(a,b)$ is not an inner product space.

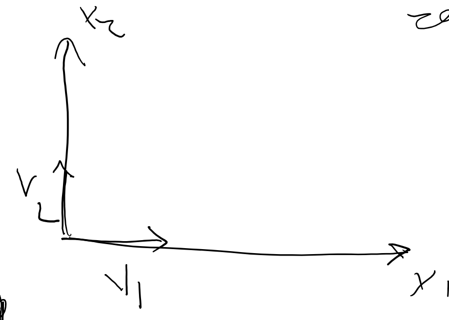
Coordinates and coordinate transformation:

Linear independence:

$\underset{}{v_1, v_2, \ldots, v_n \in \underset{\text{vector space}}{V}}$ are called linearly independent if for $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ $\quad \alpha_1 v_1 + \ldots + \alpha_n v_n = 0 \implies \alpha_1, \ldots, \alpha_n = 0$ must be zero







linearly independent

$2v_1 - v_2 = 0$

not independent

Basis $e = \{e_1, \ldots, e_n\}$ for vector space V has the following properties

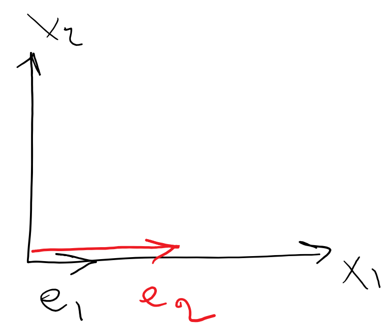
① $e_1, \ldots, e_n$ are linearly independent

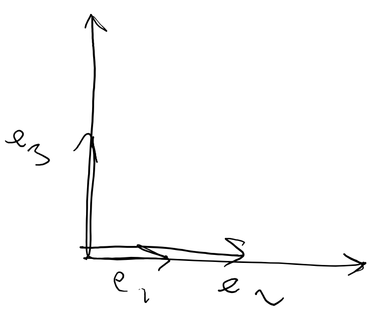
② Any vector v can be expressed in term of the basis

n: dimension of vector space

$v = v_1 e_1 + \ldots + v_n e_n$ $\quad (v_1, \ldots, v_n)$ is coordinate of v w.r.t. e coordinate system

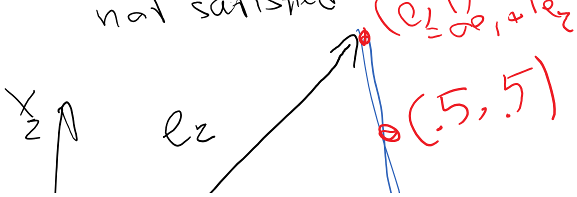
Examples for basis



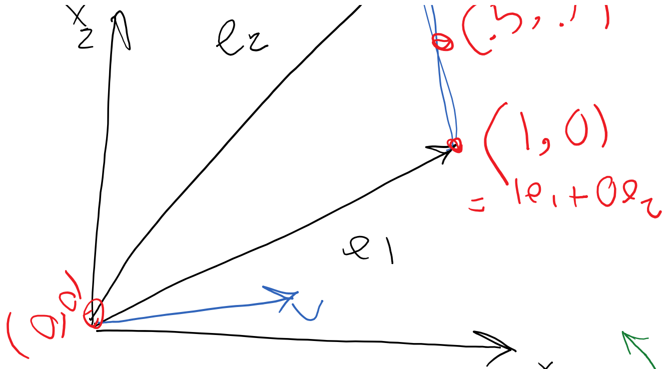


①&② are not satisfied

property 1 is violated



natural coordinate for triangle (for all simplices)

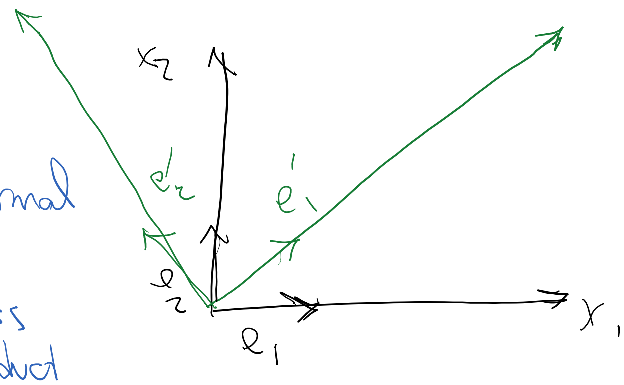


$v = v_1 e_1 + v_2 e_2$

for triangle (for all simplices)

$|e_1|, |e_2| \neq 1$

$e_1 \cdot e_2 \neq 0$



$|e_i| = 1$

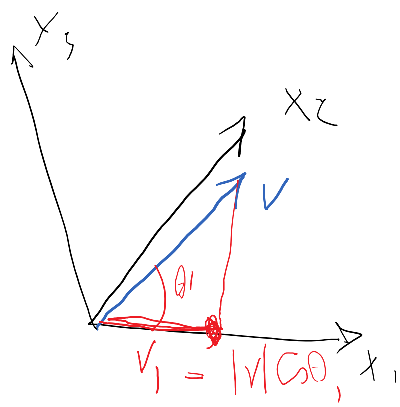
$e_i \cdot e_j = \delta_{ij}$

Orthonormal basis

can only discuss this form inner product vector spaces

Interpretation of coordinates of a vector in a given orthonormal basis

$V_i = ?$ in an orthonormal basis



$V = V_i e_i \quad \cdot e_j$

$V \cdot e_j = (V_i e_i) \cdot e_j$

$= V_i \underbrace{(e_i \cdot e_j)}_{\text{orthonormality of the basis}} \quad = \delta_{ij}$

$= V_i \delta_{ij} = V_j$

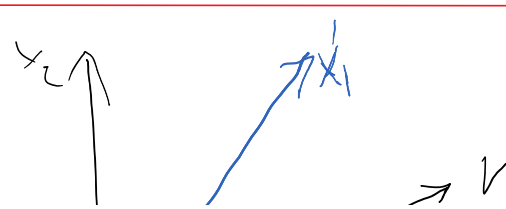
$$\boxed{V_i = V \cdot e_i = |V| \cos\theta_i \, |e_i| = P_{e_i}(V)}$$

$|e_i| = 1$

Coordinate transformation:

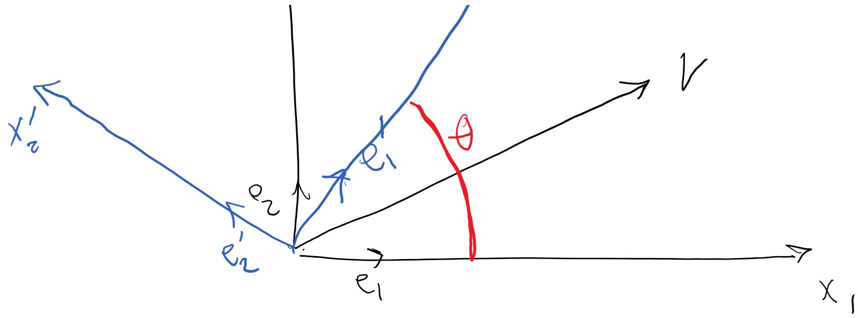
$v = v_1 e_1 + v_2 e_2 \qquad (v_1, v_2)$

$v = v_1' e_1' + v_2' e_2' \qquad (v_1', v_2')$



$v = v_1 e_1 + v_2 e_2$ $\quad (v_1, v_2)$

$v = v_1' e_1' + v_2' e_2'$ $\quad (v_1', v_2')$



$\{e_1, e_2\}$ & $\{e_1', e_2'\}$ are both orthonormal basis.

$$(v_1, v_2) \Longrightarrow (v_1', v_2') = ?$$

In general (2D, 3D) we can write

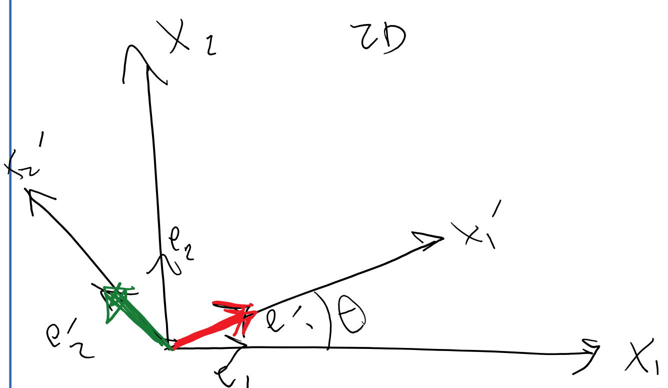
$$\boxed{e_i' = Q_{ij} e_j}$$

Q (TAM 551 λ is used for it) is called coordinate transformation matrix

$$\left(e_i' = Q_{ij} e_j\right) \cdot e_k \qquad e_i' \cdot e_k = (Q_{ij} e_j) \cdot e_k = Q_{ij} \underbrace{e_j \cdot e_k}_{\delta_{jk}} = Q_{ij}\, \delta_{jk} \; (j \to k)$$

$$e_i' = Q_{ij} e_j \longrightarrow Q_{ij} = e_i' \cdot e_j$$

component j of e_i' in coordinate system $\{e_1, \dots, e_n\}$



$$Q = \begin{bmatrix} e_1' \\ \hline e_2' \\ \hline e_3' \end{bmatrix}$$

unit vectors of ( )' expressed in ( ) coordinate system

3D

Q for 2D figure above

$$Q = \begin{bmatrix} \overbrace{\cos\theta \quad \sin\theta}^{e_1'} \\ -\sin\theta \quad \cos\theta \end{bmatrix}$$

$$e_i' = Q_{ij}\, e_j \qquad Q_{ij} = e_i' \cdot e_j \qquad \Longrightarrow R_{ij} = Q_{ij}$$

$e'_i = Q_{ij}\, e_j$ $\qquad Q_{ij} = e'_i \cdot e_j$

$e_j = \underbrace{R_{ji}}_{?}\, e'_i$ $\xrightarrow{\text{Same trick}}$ $R_{ji} = e_j \cdot e'_i$ $\Bigg\} \Rightarrow R_{ji} = Q_{ij}$

$(\ )\cdot e'_k$

$$\boxed{\begin{aligned} e'_i &= Q_{ij}\, e_j \\ e_j &= Q_{ij}\, e'_i \\ &= Q^T_{ji}\, e'_i \end{aligned}}$$

$\{e'\} = Q\{e\} \Longrightarrow \left[\begin{aligned} \{e\} &= Q^{-1}\{e'\} \\ \{e\} &= Q^{T}\{e'\} \end{aligned}\right] \longrightarrow$

$$\boxed{Q^{-1} = Q^{T}}$$

Q is an orthognal matrix

$$Q^T Q = Q Q^T = \left(Q^{-1} Q = Q Q^{-1}\right) = I$$

$$Q Q^T = \underbrace{\begin{bmatrix} e'_1 \\ \hline e'_2 \\ \hline e'_3 \end{bmatrix}}_{Q} \begin{bmatrix} e'_1 & \Big| & e'_2 & \Big| & e'_3 \end{bmatrix} = \begin{bmatrix} e'_1 \cdot e'_1 & e'_1 \cdot e'_2 & e'_1 \cdot e'_3 \\ e'_2 \cdot e'_1 & e'_2 \cdot e'_2 & e'_2 \cdot e'_3 \\ e'_3 \cdot e'_1 & e'_3 \cdot e'_2 & e'_3 \cdot e'_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

because $e'_i \cdot e'_j = \delta_{ij}$

What is the use of Q?

$\left.\begin{aligned} v &= v_j\, e_j \\ e_j &= Q_{ij}\, e'_i \end{aligned}\right\} \Longrightarrow \left[\begin{aligned} v &= \underbrace{(Q_{ij}\, v_j)}\, e'_i \\ &= v'_i\, e'_i \end{aligned}\right] \longrightarrow$

$$\boxed{v_i' = Q_{ij} v_j \qquad [v]' = Q[v]}$$

$$\boxed{v_j = Q_{ij} v_i' \qquad [v] = Q^T [v]'}$$

In general for m^{th} order tensor

$$\boxed{T'_{i_1 \dots i_m} = Q_{i_1 j_1} Q_{i_2 j_2} \dots Q_{i_m j_m} T_{j_1 j_2 \dots j_m}}$$