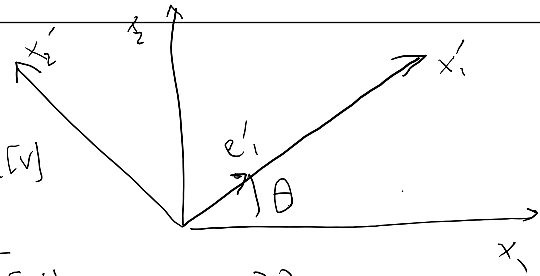
2019/09/11

$$\begin{cases} e_i' = Q_{ij}\, e_j \\ v_i' = Q_{ij}\, v_j \end{cases} \qquad Q = \left[\, e_1' \,\middle|\, e_2' \,\middle|\, e_3' \,\right] \quad [e]Q[e]$$

in ( ) system $[v'] = Q[v]$



$$e_j = Q_{ij}\, e_i' \qquad Q = \begin{bmatrix} e_1 \\ \hline e_2 \\ \hline e_3 \end{bmatrix} \qquad [e] = Q^T [e']$$

$$v_j = Q_{ij} V_i' \qquad\qquad [v] = Q^T [v']$$

in ' system

2D

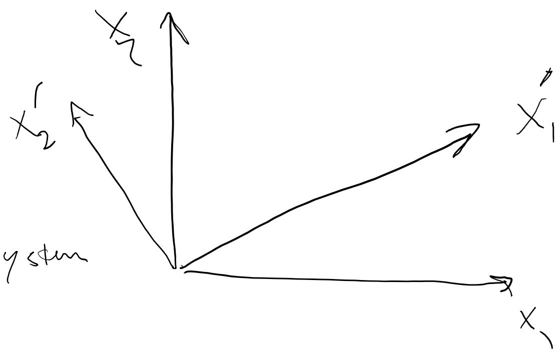
$$Q = \begin{bmatrix} c & -s \\ +s & c \end{bmatrix} \qquad c = \cos\theta \quad s = -\sin\theta$$

Scalars $\lambda \in \mathbb{R}$ is a scalar

$T(x)$

↓

temperature is a scalar



Scalars don't change from one coordinate system to another

Example:

Inner product is a scalar

— Part A $\quad a = a_i e_i \quad b = b_i e_i \qquad \boxed{a \cdot b = a_i b_i}$ (★) $\quad \{e_i\}$ orthonormal coordinate system

want to show that IF we had used (★) as definition of $\cdot$ then $\cdot$ is a scalar.

$$a \cdot b = (a_i e_i) \cdot (b_j e_j) = a_i b_j \underbrace{(e_i \cdot e_j)}$$

$$= \begin{cases} a_i b_j \underbrace{\delta_{ij}} = a_i b_i & \text{case 1 orthonormal } \{e_i\} \\ a_i b_j \underbrace{g_{ij}}_{\text{metric matrix}} & \text{case 2 not covered here general coordinates} \end{cases}$$

for a coordinate system

metric matrix for a coordinate system

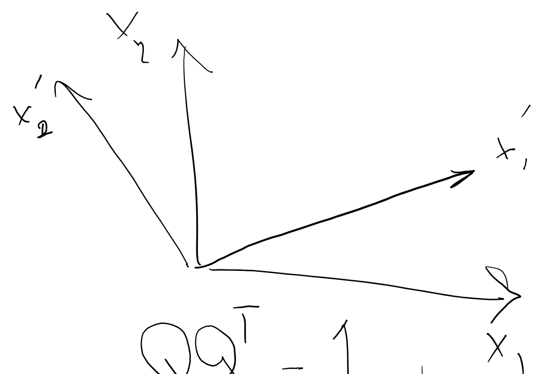
$$a \cdot b = [a_1 \cdots a_n]\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = a_i b_i \qquad \text{orthonormal } \{e_i\} \qquad g = I$$

$$= [a_1 \cdots a_n]\begin{bmatrix} g_{11} & \cdots & g_{1n} \\ & \ddots & \\ g_{n1} & & g_{nn} \end{bmatrix}\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = a_i g_{ij} b_j \qquad g_{ij} = e_i \cdot e_j$$

nonorthonormal $\{e_i\}$

- Part B

$a_i b_i$ definition is coordinate independent



$$a_i b_i = Q_{mi} Q_{ni} a'_m b'_n$$

$$a_i = Q_{mi} a'_m \qquad b_i = Q_{ni} b'_n$$

$$= \underbrace{Q_{mi} (Q^T)_{in}} a'_m b'_n$$

$$(QQ^T)_{mn} a'_m b'_n$$

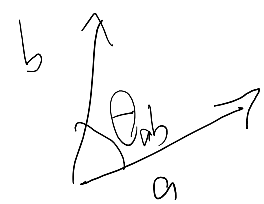
$$\underbrace{\delta_{mn} a'_m b'_n} =$$

$$QQ^T = 1_{matrix}$$

$$\Longrightarrow \boxed{a_i b_i = a'_m b'_m}$$

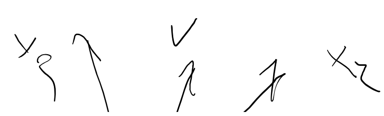
The reason this works nicely, we defined inner product independent of vector components

$$a \cdot b = |a||b| \cos \theta_{a,b}$$



A number that is not a scalar

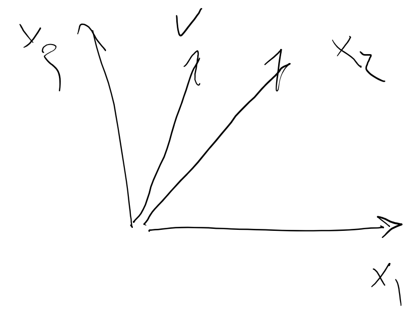
$v - V_i e$: ________



A number that is ...

$V = V_i e_i$

$$\ell_p(V) := \sqrt[p]{\sum_{i=1}^{3} |V_i|^p}$$

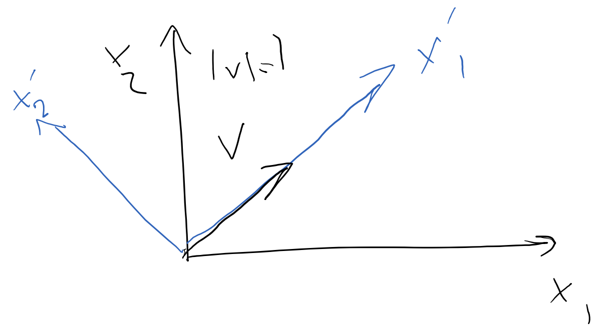


$$\boxed{\ell_1(V) = |V_1| + |V_2| + |V_3|}$$ Not a scalar!

$\ell_2(V) = \sqrt{|V_1|^2 + |V_2|^2 + |V_3|^2} = \sqrt{V \cdot V} = |V|$ magnitude of V scalar

$\ell_1(V) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$ in $\{e_1, e_2\}$

$\ell_1(V) = 1 + 0 = 1$



Tensors			coordinate transformation
0th order	λ	scalar	λ
1st "	V	vector	$V_i' = Q_{ij} V_j$
2nd "	A		$A_{ij}' = Q_{im} Q_{jn} A_{mn}$
⋮		
4th order	C e.g. elasticity tensor		$C_{ijkl}' = Q_{im} Q_{jn} Q_{kp} Q_{lq} C_{mnpq}$

2nd order tensors or

Linear operators (functions)

A linear operator L, is a function from vector space V to vector space W that satisfies the following two properties

$$L \quad V \to W$$

... Combined

$L: V \to V$

$\forall v_1, v_2 \quad L(v_1+v_2) = L(v_1) + L(v_2)$

$\forall v_1 \in V\ \lambda \in \mathbb{R} \quad L(\lambda v_1) = \lambda L(v_1)$

Combined

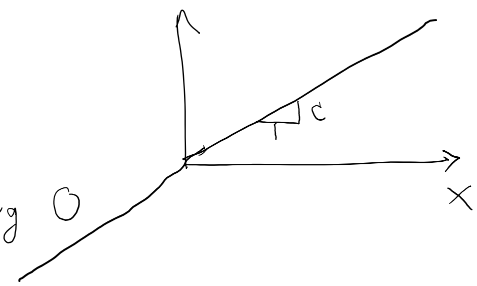
$L(\alpha v_1 + \beta v_2) = \alpha L(v_1) + \beta L(v_2)$

or

$L(v_1 + \beta v_2) = L(v_1) + \beta L(v_2)$

Example: 1D $\quad \mathbb{R} \to \mathbb{R}$

$\forall x \quad f(x) = f(\underset{\text{scalar}}{x} \cdot 1) = x\underbrace{f(1)}_{c} = cx$



Linear function in $\mathbb{R}^n$ they are planes passing throug 0

For any Linear Function $L(0) = 0$

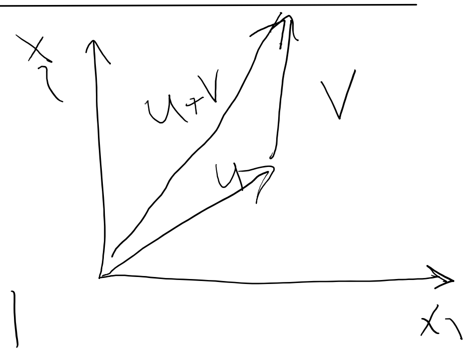
$0 + L(0) = L(0+0) = L(0) + L(0) \implies \boxed{L(0) = 0}$

Example 2

$L(u) = |u| \quad R^2 \to R$

is this linear $\quad L(\vec{0}) = 0$

But this is not a linear function



$L(u+v) = |u+v| \cancel{=} \underbrace{|u|}_{L(u)} + \underbrace{|v|}_{L(v)}$

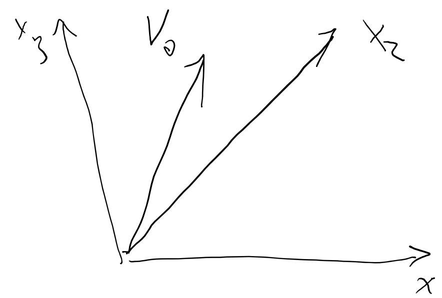
$\leq$

Not a linear function

Example 3 $\quad v_0 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$L(u) = v_0 \cdot u \quad \underbrace{R^2}_{V} \longrightarrow \mathbb{R}$

inner product



$[v_1]$

inner product

$$L(u) = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = u_1 + 2u_2 + 3u_3$$

$$L(u + \beta v) = (u + \beta v)_1 + 2(u + \beta v)_2 + 3(u + \beta v)_3 =$$

$$(u_1 + 2u_2 + 3u_3) + \beta(v_1 + 2v_2 + 3v_3) = L(u) + \beta L(v)$$

linear

In fact all linear operators from $V \longrightarrow \mathbb{R}$ can be written in the form of inner product

$\exists v_a \in V \quad L(v) = v_a \cdot v$

There is one remaining point to consider before proceeding on to second-order tensors. That is, the inner product allows us to interpret a (Euclidean) vector as a linear operator that maps a vector into a real number (scalar). In fact, this is the defining property of a *first-order tensor*, and vectors are indeed first order tensors. To illuminate this point, let $\mathcal{V}$ be the set of all vectors in some Euclidean point space $\mathcal{E}$. Now consider a specific vector $\bar{\mathrm{a}} \in \mathcal{V}$, where the overbar indicates that we hold $\bar{\mathrm{a}}$ fixed. We can define a function $f_{\bar{\mathrm{a}}}$ that maps a vector into a scalar by taking the inner product of $\bar{\mathrm{a}}$ and any vector $\mathbf{b} \in \mathcal{V}$. That is,

$$f_{\bar{\mathrm{a}}}(\mathbf{b}) \equiv \bar{\mathrm{a}} \cdot \mathbf{b}.$$

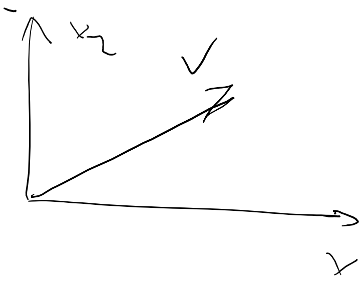
A review of the properties of the inner product shows that $f_{\bar{\mathrm{a}}}$ is indeed a linear operator. In fact, the *Riesz representation theorem* states that *every* linear function on $\mathcal{V}$ to $\Re$ can be represented in this fashion (by varying our choice of the fixed vector $\bar{\mathrm{a}}$)! We use a similar approach in the next section to define second-order tensors as a special class of linear operators.

Related to this topic (suitable for a term project)
Concept of vectors, covectors, dual basis, ..

$V = v^i \underline{e_i}$

$L(v) = a \cdot V$

$a = a^i e_i$



e_i's are orthonormal

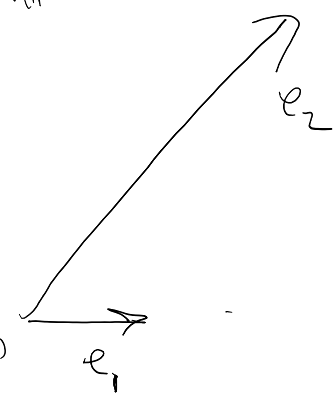
$a = a^i e_i$ orthonormal

$$L(v) = (v^i e_i)(a^j e_j) = v^i a^j \underbrace{(e_i \cdot e_j)}_{\text{metric matrix}}$$

$$= \begin{cases} \delta_{ij} & e_i \text{ orthonormal} \\ g_{ij} & \text{" Not "} \end{cases}$$

$$L(v) = v^i a^j g_{ij}$$

Dual basis $e^i(e_j) = \delta^i_j$



If the basis is orthonormal e^i is the same as e_i

. . . .

We can represent linear operators $V \to \mathbb{R}$ by dual basis

$$L = l_i e^i$$

$$L(v) = (l_i e^i)(v^j e_j) = l_i v^j \underbrace{(e^i e_j)}_{\delta^i_j} = l_i v^i$$

In contrast if we had written L in terms basis

$L = l^i e_i$ $\quad$ $L(v) = l^i v^j g_{ij}$

δ_{ij} $\quad$ $F^i_{\ j}$ $\quad$ T_{ij} $\quad$ $T^i_{\ j}$ $\quad$ T^{ij} $\quad$ $T^i_{\ j}$

For the rest of the course we use orthonormal basis, so indices up or down don't make any difference

Linear functions (operators) themselves form a vector space

space of all linear operators from $\underline{\mathbb{V}}$ $V \longrightarrow W$

$S, T, U \in \bar{V} : V \to W$

$\lambda, \mu \in \mathbb{R}$

addition properties

$S + T = T + S$

$S + (T + U) = (S + T) + U$

$S + \underline{\underline{O}} = S$

scalar product properties

$(\lambda \mu) S = \lambda (\mu S)$

$\lambda (S + T) = \lambda S + \lambda T$

$(\lambda + \mu) S = \lambda S + \mu S$

$\underline{1} S = S$

$S + T = ?$

$\lambda S = ?$

$O = ?$

$(S + T)(u) := S(u) + T(u)$

$(\lambda S)(u) := \lambda (S(u))$

$O(0_V) = O_W$

Linear operator zero

You need to show that in fact if $\lambda \in \mathbb{R}$, S, T $V \xrightarrow{\text{linear}} W$

so are $S + T$

λS

$$(S+T)(u+\beta v) = (S+T)(u) + \beta (S+T)(v)$$

$$= S(u+\beta v) + T(u+\beta v) \qquad \text{def. } S+T$$

$$= (Su+\beta Sv) + (Tu+\beta Tv) \qquad S, T \text{ are linear}$$

$$= (Su+Tu) + \beta(Sv+Tv) \qquad \text{3 properties of vector space } W$$

$$= (S+T)(u) + \beta(S+T)(v) \qquad \text{def } S+T$$

Dyadic product of two vectors:

$$u \cdot v = \underbrace{[u_1 \; \cdots \; u_n] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}}_{} = u^T v = u_i v_i$$

$$u \otimes v = \underset{n\times 1}{\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}} \underset{1\times n}{[v_1 \; \cdots \; v_n]} = u v^T = \begin{bmatrix} u_1 v_1 & u_1 v_2 & \cdots & u_1 v_n \\ u_2 v_1 & & & \\ \vdots & & & \\ u_n v_1 & \cdots & \cdots & u_n v_n \end{bmatrix}$$

motivation

Definition of dyadic product:

for vectors $u, v \in V$ we define dyadic product as an operator from $V \to V$ defined as

$$\forall w \in V \qquad \underbrace{(u \otimes v)(w)}_{\text{vector}} = \overbrace{u \, (\underbrace{v \cdot w}_{\text{scalar}})}^{\text{vector in } V}$$

we need to show that $u \otimes v$ is in fact a linear operator

$$(u\otimes v)(\alpha a + \beta b) = u\{v \cdot (\alpha a + \beta b)\} \qquad u, v, a, b \in V, \quad \alpha, \beta \in \mathbb{R}$$

$$= u\{\alpha (v \cdot a) + \beta (v \cdot b)\} \qquad \text{dist. \& homog. of inner product}$$

$$= u\{\alpha(v\cdot a) + \beta(v\cdot b)\}$$ dist. & homog. of inner product

$$= \alpha(u(v\cdot a)) + \beta(u(v\cdot b))$$ dist. & scalar product properties of V

$$= \alpha(u\otimes v(a)) + \beta((u\otimes v)(b))$$

$L(\alpha a+\beta b) = \alpha L(a) + \beta L(b)$ $\quad L = u\otimes v \quad$ so $u\otimes v$ is in fact a linear operator

---

motivation

$$e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix} \qquad e_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$e_1\otimes e_3 = \begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}0 & 0 & 1\end{bmatrix} = \begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix} \qquad e_1\otimes e_3 = (e_3\otimes e_1)^T$$

$$e_3\otimes e_1 = \begin{bmatrix}0\\0\\1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\end{bmatrix} = \begin{bmatrix}0&0&0\\0&0&0\\1&0&0\end{bmatrix}$$

$u\otimes v \neq v\otimes u$ in fact as we'll see later

$$u\otimes v = (v\otimes u)^T$$

Motivation

$$T = \begin{bmatrix}T_{11} & T_{12} & T_{13}\\ T_{21} & T_{22} & T_{23}\\ T_{31} & T_{32} & T_{33}\end{bmatrix} = T_{11}\underbrace{\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}}_{e_1\otimes e_1} + T_{12}\underbrace{\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}}_{e_1\otimes e_2} + T_{13}\underbrace{\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}}_{e_1\otimes e_3} + \dots + T_{33}\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$$

$$= T_{11}\, e_1\otimes e_1 + T_{12}\, e_1\otimes e_2 + \dots$$

$$T_{33}\, e_3\otimes e_3$$

$$\boxed{T = T_{ij}\, e_i \otimes e_j}$$

are the basis for 2nd order tensors
same way e_i are basis for vectors

$$v = v_i e_i$$

---

define $T_{ij} = e_i\left(T e_j\right)$

show $T = T_{ij}\, e_i \otimes e_j$

Dual basis (F) $T = T_{ij}\, e^i \otimes e^j = T^{ij}\, e_i \otimes e_j$

$= T_i{}^j\, e^i \otimes e_j = T^i{}_j\, e_i \otimes e^j$

$g = g_{ij}\, e^i \otimes e^j$

$F = F^i{}_j\, e_i \otimes e^j$