Higher order tensors:

Motivation

In 1D stress is related to strain through elastic modulus:

\[ \sigma = E \varepsilon \]

\[ \varepsilon = \varepsilon_0 + x \]

In 2D and 3D, stress and strain are second order tensors.

\[ \varepsilon = \frac{1}{2} ( \nabla u + \nabla u^T ) \]

\[ \sigma = C \varepsilon \]

In indicial notation for this equation,

\[ \varepsilon = T u \]

\[ v_i = \frac{\partial u_j}{\partial x_i} \]

\[ \sigma = C \varepsilon \]

\[ C_{ijkl} e_i e_j e_k e_l \]

\[ e_i = Q_{mi} e^m \]

\[ e^i = Q^{mi} e_m \]

4th order and higher order coordinate transformation:

\[ C = C_{ijkl} e_i e_j e_k e_l \]

\[ C'' = C_{mnpq} e_i e_j e_k e_l \]

A four-indexed array that follows this transformation rule is a 4th order tensor.

Polyads as generalization of dyadic product
Components of a tensor:

\[ \mathbf{V} = \{ v_i \} \]

\[ \mathbf{V} = \mathbf{v}_i e_i \]

\[ \mathbf{V} = \mathbf{v}_i e_i \]

\[ T \equiv T_{ij} e_i \otimes e_j \]

\[ \mathbf{T} \equiv \mathbf{T}_{ij} \mathbf{e}_i \otimes \mathbf{e}_j \]

See definition 46 for components of m'th tensor

Theorem 84 for equation (*)

Theorem 88 for coordinate transformation of m'th tensor

Tensor product in general:

\[ \mathbf{T} \otimes \mathbf{S} = \mathbf{T} \mathbf{S} \]

\[ \mathbf{T} \mathbf{S} = \mathbf{T}_{ij} \mathbf{S}_{kl} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l \]

Identity matrix for higher order tensors

\[ \mathbf{I} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \]

See Definition 50 for more discussion on contraction
Identity matrix for higher order tensors

\[ \mathbf{I} v = v \]
\[ \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]
\[ \mathbf{I}^m = \mathbf{I} \begin{bmatrix} 2^m \\ m \\ 1 \end{bmatrix} \]
\[ \mathbf{I}_{ijkl} = ? \]

1.13 The vector/cross/exterior product of vectors:

\[ A = |u|v| \sin \theta \]
\[ u \cdot v = |u||v| \cos \theta \]
\[ u \times v = ? \]

We use the 3rd order alternating tensor

\[ \mathbf{\varepsilon} = \epsilon_{ijk} e_i \times e_j \times e_k \]

\[ u \times v = (\mathbf{\varepsilon} \cdot v) u \]
\[ u \cdot v \rightarrow \text{scalar} \]

\[ u \times v = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \]

\[ u \times v = \varepsilon_{ijk} u_i v_j e_k \]

Q: Is cross product associative?
Theorem 93  The vector product is not associative:

\((u \times v) \times w \neq u \times (v \times w)\),

indeed

\[(u \times v) \times w = (u \cdot w)v - (v \cdot w)u,\]
\[u \times (v \times w) = (u \cdot w)v - (u \cdot v)w.\]

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**Triple Product:**

\[h = |w| \cos \psi\]

\[\mathbf{V} = \mathbf{A} \cdot \mathbf{h} = \left( \frac{|w| \cos \psi}{|A|} \right) |w| \cos \psi\]

\[= |u \times v| \cos \psi \cos \left(\mathbf{u} \times \mathbf{v}, \mathbf{w}\right)\]

\[= (u \times v) \cdot \mathbf{w}\]

\[= \begin{vmatrix}
\hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\
\mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \\
\mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3
\end{vmatrix}
= \begin{vmatrix}
\mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3 \\
\mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3
\end{vmatrix}\]

\[= \mathbf{V} \cdot \mathbf{w}\]

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2nd order tensors that we have not covered yet:

1. Orthogonal
2. Skew symmetric

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Continuum Page 4
2nd order tensors that we have not covered yet:

1. Orthonormal
2. Skew symmetric
3. Symmetric
4. Positive definite

Orthonormal or orthogonal tensors:

For second order orthogonal tensors we have:

\[ T^T = T^{-1} \]

\[ T T^k = I \]

\[ T^k = T \]

\[ T T^k = I \]

What does an orthogonal tensor represent?
Orthogonal tensors are rotations plus possibly 1 reflection.
Orthogonal tensors preserve angle, magnitude, inner product, and distances

The following statements are equivalent:

1. \( T \) is \( \text{Orthogonal} \)

2. \( \forall u, v \) \( \langle T u, T v \rangle = u \cdot v \) preserves

3. \( \forall u \) \( |Tu| = |u| \) magnitude

4. \( \forall u, v \) \( |Tu - T v| = |u - v| \)

\[ T u . T v = T^T u . v \]

\[ \text{when } T^T T = I \] \( T \) orthogonal

1 \( \iff \) 2

2 \( \implies \) 3

\[ T u . T v = T u . v \]

\[ \implies \]

\[ \text{square all these } \]

\[ \implies \]

\[ T (u + v) = |u + v| \]

3 \( \implies \) 2

\[ |Tu| = |u| \]

\[ |Tv| = |v| \]

\[ |T(u + v)| = |u + v| \]

4 \( \implies \) 2

\[ |Tu - T v| \]

\[ |T u - T v| \]

\[ |T u - T v| \]

\[ |Tu - T v| \]

\[ |T u - T v| \]

\[ |T u - T v| \]

\[ |T u - T v| \]

\[ |T u - T v| \]
Orthogonal tensors also preserve angle? Why?

\[
G_{\theta_{Tu,TV}} = \frac{Tu \cdot Tv}{|Tu||Tv|}
\]

\[
= \frac{u \cdot v}{|u||v|}
\]

\[
= G_{\theta_{u,v}}
\]

Skew-symmetric tensors
Well, they too also represent rotation!

\[
W = -W^t
\]

\[
W_{ij} = -W_{ji}
\]

\[
\Rightarrow u \cdot W u = 0
\]

Definition of axis of a second order skew-symmetric tensor

\[
W = \begin{bmatrix}
0 & -W_{12} & W_{13} \\
-W_{12} & 0 & W_{23} \\
-W_{13} & -W_{23} & 0 \\
\end{bmatrix}
\]

\[
W_{ij} = -W_{ji}
\]

\[
\Rightarrow \theta_{Wu,u} = 0
\]
\[ W = \begin{bmatrix} -W_{12} & 0 & W_{23} \\ W_{31} & -W_{23} & 0 \end{bmatrix} \]

\[ W_u = \begin{bmatrix} 0 & w_1 & w_2 \\ -w_2 & 0 & w_3 \\ w_3 & -w_1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} w_{12} u_2 - w_{31} u_3 \\ w_{21} u_1 - w_{32} u_3 \\ w_{31} u_1 - w_{23} u_3 \end{bmatrix} \]

\[ \omega = \alpha \times W \]

\[ W = \alpha \times \omega \]

1-1 correspondence between vectors & skew-sym. 2nd order tensors

\[ \omega_i = \frac{1}{2} \epsilon_{ijk} w_{jk} \]

\[ W_{ij} = -\epsilon_{ijk} w_{ik} \]

Rotation:

\[ u_\perp = u_{11} + u_{12} + u_{13} \]

\[ \text{small rotation change} \]
Look at this from the top

Real change of location

\[ \Delta u = Q_u - u \]

\[ \alpha = \theta \alpha \]

\[ u = \theta a \]

\[ u \times u \]

Magnitude

[Diagram with vectors and angles]

\[ w \mathbf{u} \sim Q_u - u \]