2019/09/18

Wednesday, September 18, 2019 10:44 AM

Higher order tensors:

Motivation

In 1D stress is related to strain through elastic modulus:

$$\sigma = E\varepsilon \qquad \varepsilon = u_{,x}$$

In 2D and 3D, stress and strain are second order tensor

$$\sigma_{ij} \qquad \varepsilon = \frac{1}{2}(\nabla u + \nabla^T u)$$

1D

$$\sigma = E\varepsilon \implies \underbrace{\sigma}_{\text{2nd order tensor}} = \underbrace{\mathbb{C}}_{\text{4th order elasticity tensor}}\,\varepsilon$$

linear solid

$$1 = 2 - 1$$

Inidicial notation for this equation

$$\underbrace{v = Tu}_{\text{vectors}} \qquad v_i = T_{ij} v_j$$

contraction

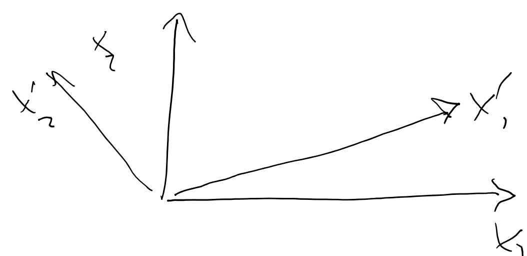
$$2 = 4 - 2$$

$$\sigma = \mathbb{C}\varepsilon \qquad \sigma_{ij} = \mathbb{C}_{ijkl}\,\varepsilon_{lk}$$

$$\mathbb{C}_{ijkl}\,\underbrace{\varepsilon_{kl}}_{\varepsilon_{lk}}$$

4th order and higher order coordinate transformation:

$$\left.\begin{aligned} \mathbb{C} &= \mathbb{C}_{ijkl}\, e_i \otimes e_j \otimes e_k \otimes e_l \\ e_i &= Q_{mi}\, e'_m \\ &\vdots \\ e_l &= Q_{ql}\, e'_q \end{aligned}\right\} \implies \mathbb{C}'_{mnpq} = Q_{mi} Q_{nj} Q_{kp} Q_{lq} \mathbb{C}_{ijkl}$$



A four-indexed array that follows this transformation rule is a 4th order tensor

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Polyads as generalization of dyadic product

$$(\underbrace{u_1 \otimes u_2 \otimes \cdots \otimes u_n}_{n^{th}\ \text{order tensor}})\, w = (u_n \cdot w)\,(u_1 \otimes u_2 \otimes \cdots \otimes u_{n-1})$$

similar to dyadic product $(u_1 \otimes u_2)w = (u_2 \cdot w)\, u_1$

Components of a tensor

0: λ

1 vectors: $V = V_i e_i$, $V_i = V \cdot e_i$

2nd: $T = T_{ij}\, e_i \otimes e_j$, $T_{ij} = e_i \cdot (T e_j)$

m^{th} order: (*) $T = T_{i_1 \cdots i_m}\, e_1 \otimes e_2 \otimes \cdots e_m$

$$T_{i_1 \cdots i_m} = e_{i_1} \cdot \Big[\big(\underbrace{(\overset{m}{T} e_m)}_{\substack{m-1 \\ \text{order}}} e_{m-1} \big)_{m-2\ \text{order}} \cdots e_{i_2} \Big]$$

See definition 46 for components of m'th tensor

Theorem 84 for equation (*)
Theorem 88 for coordinate transformation of m'th tensor

Tensor product in general:

$\overset{m}{T}$: m^{th} order tensor

$\overset{n}{S}$: n^{th} " "

$\underbrace{TS}_{m-n}$ order tensor

$$TS = \overset{m}{T}_{i_1 i_2 \cdots i_{(m-n+1)} \cdots i_m}\ \overset{n}{S}_{j_1 j_2 \cdots j_n}$$

$i_m = j_1$, j_2, j_n → contraction

Reduces the order of tensor in general

$$\boxed{(TS)_{i_1 \cdots i_{m-n}} = \overset{m}{T}_{i_1 \cdots i_{m-n}\, j_n \cdots j_1}\ \overset{n}{S}_{j_1 j_2 \cdots j_n}}$$

See Definition 50 for more discussion on contraction

Ex: $\underset{\text{2nd order}}{T_{ij}} \longrightarrow \underset{\text{trace}}{T_{ii}}$

Identity matrix for higher order tensors

Identity matrix for higher order tensors

$$\mathbf{I}v = v \qquad [\mathbf{I}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\overset{m}{\mathbf{T}} = \overset{2m}{\mathbf{I}}\ \overset{m}{\mathbf{T}} \qquad \overset{4}{\mathbf{I}}_{ijkl} = ?$$

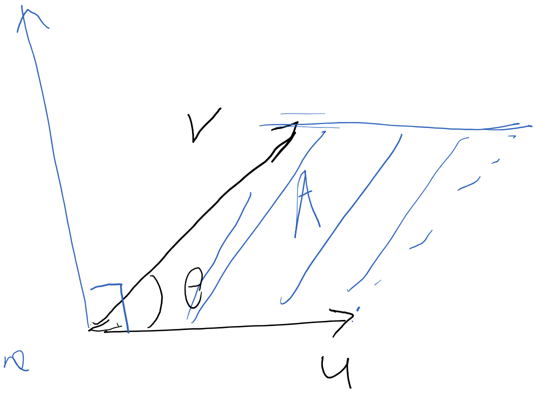
1.13 The vector/cross/exterior product of vectors:

$A = |u||v| \sin\theta$

$u \cdot v = |u||v| \cos\theta$

$|u \times v| = A$

$u \times v$ is normal to the plane of u, v



$u \times v = ?$

we use the 3rd order alternating tensor

$$\overset{3}{\mathbf{E}} = \epsilon_{ijk}\, e_i \otimes e_j \otimes e_k$$

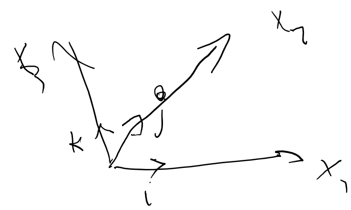
$$u \times v = \underbrace{\underbrace{(\overset{3}{\mathbf{E}}\, v)}_{\text{2nd order tensor}}\, u}_{\text{vector}}$$

$u \cdot v \Rightarrow$ scalar

$$\boxed{u \times v = \det \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}}$$

$\hat{i}, \hat{j}, \hat{k}$ are unit vectors

e_1, e_2, e_3



$$\boxed{u \times v = \epsilon_{ijk}\, u_i v_j e_k}$$

Q: Is cross product associative?

$u \times (v \times w) = (u \times v) \times w$? No

Theorem 93 *The vector product is* not *associative:*

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} \neq \mathbf{u} \times (\mathbf{v} \times \mathbf{w}),$$

indeed

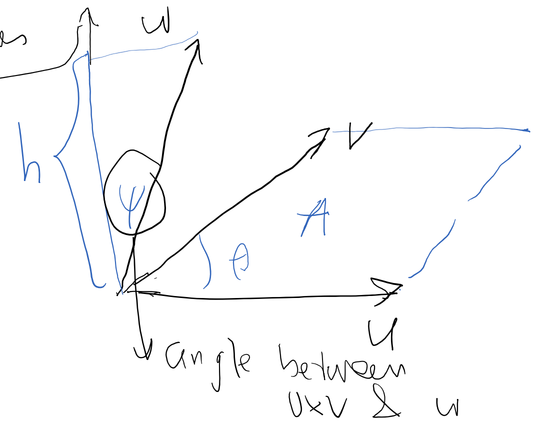
$$\begin{aligned}(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} &= (\mathbf{u} \cdot \mathbf{w})\,\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\,\mathbf{u}, \\ \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) &= (\mathbf{u} \cdot \mathbf{w})\,\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\,\mathbf{w}.\end{aligned}$$

Triple Product:

used for computing volumes

direction of $u \times v$

$h = |w| \cos\psi$



$$V = A \cdot h = \underbrace{\big(\underbrace{|u||v|\sin\theta}_{A}\big)}\ |w| \cos\psi$$

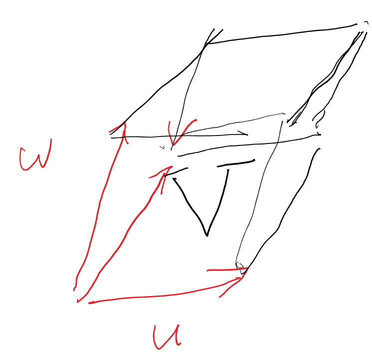
$$|u \times v|\ |w| \cos\psi_{(u\times v,\, w)}$$

$$a \qquad b \qquad \cos\psi_{(a,b)}$$

$$= (u \times v) \cdot w$$

$$= \begin{vmatrix} e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \cdot (w_1 e_1 + w_2 e_2 + w_3 e_3)$$

$$\boxed{(u \times v) \cdot w = \det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} = V}$$



2nd order tensors that we have not covered yet:

1. Orthonormal
2. Skew symmetric

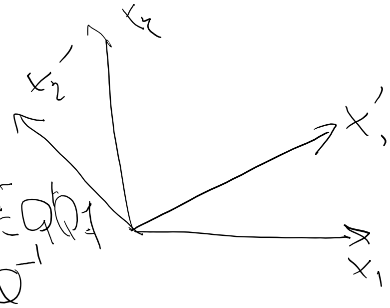
Xv

2nd order tensors that we have not covered yet:

1. Orthonormal
2. Skew symmetric
3. Symmetric
4. Positive definite

For Matrix Q we had $QQ^t = Q^tQ = I$

$Q^t = Q^{-1}$



Orthonormal or orthogonal tensors:

For second order Orthogonal tensors we have

$$\boxed{\begin{gathered} TT^t = T^tT = I \\ T^t = T^{-1} \end{gathered}}$$

$$[T]_{ij} = \begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \left[\begin{array}{c|c|c} T_{11} & T_{12} & T_{13} \\ \hline T_{21} & T_{22} & T_{23} \\ \hline T_{31} & T_{32} & T_{33} \end{array}\right] \end{array} \begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array}$$

$r_i \cdot r_j = \delta_{ij}$

$TT^t = I$

$T^tT = I \qquad c_i \cdot c_j = \delta_{ij}$

$$T_R = \left[\begin{array}{c|c} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array}\right]$$

$\det T = \cos^2\theta + \sin^2\theta = 1$

What does an orthogonal tensor represent?
Orthogonal tensors are rotations plus possibly 1 reflection

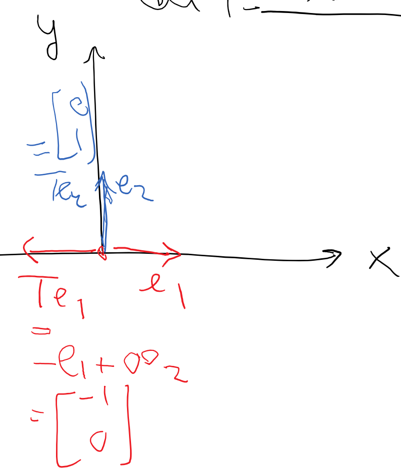
Reflection w.r.t y axis

$$T = \left[\begin{array}{c|c} T_{11} & T_{12} \\ T_{21} & T_{22} \end{array}\right]$$

c_1

$Te_1 = \begin{bmatrix} T_{11} \\ T_{21} \end{bmatrix}$

$\boxed{c_i \text{ is the image of } e_i}$



$$T = \begin{array}{c} \left[\begin{array}{c|c} -1 & 0 \\ 0 & 1 \end{array}\right] \\ \begin{array}{cc} Te_1 & Te_2 \end{array} \end{array}$$

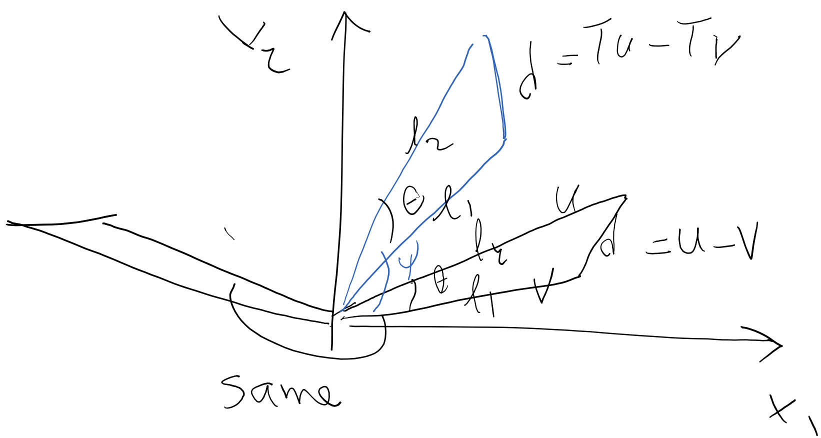
$\det T = (-1)(1)$
$= -1$

$\det Q = \begin{cases} 1 & \text{Rotation} \\ \end{cases}$

$QQ^t = I \qquad (\det Q)^2 = 1$

$QQ^t = I \implies (\det Q)^2 = 1 \qquad \det Q = \begin{cases} 1 & \\ -1 & \text{Rotation} + 1 \text{ reflection} \end{cases}$

$T_R T$



Orthogonal tensors preserve angle, magnitude, inner product, and distances

The following statements are equivalent:

1. $T \in \mathrm{Orth}\,V$
2. $\forall u, v \quad Tu \cdot Tv = u \cdot v$ ∘ preserved
3. $\forall u \quad |Tu| = |u|$ magnitude "
4. $\forall u, v \quad |Tu - Tv| = |u - v|$

$Tu \cdot Tv = T^t Tu \cdot v = u \cdot v$ when $T^t T = I$ T orthogonal

$1 \iff 2$

$2 \implies 3 \qquad Tu \cdot T\underset{u}{v} = u \cdot v \implies Tu \cdot Tu = u \cdot u \implies |Tu| = |u|$

$3 \implies 2$

$|Tu| = |u|$

$|Tv| = |v|$

$|T(u+v)| = |u+v|$

Square all these $\implies Tu \cdot Tv = u \cdot v$ ✓

$3 \to 4 \quad |T\underset{u-v}{u}| = |u|$

$|T(u-v)| = |u-v|$

$|Tu - Tv|$

$4 \to 3$

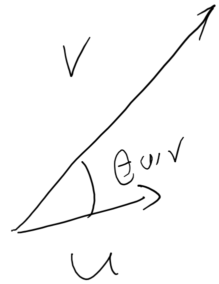
$4 \to 3 \qquad v = 0 \qquad |u| \perp |v| \qquad |Tu| = |u|$

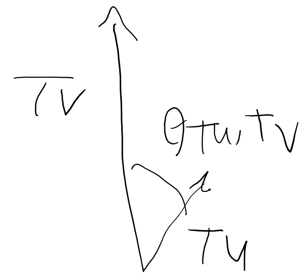
Orthogonal tensors also preserve angle?
Why?

$$\cos\Theta_{Tu,Tv} = \frac{Tu \cdot Tv}{|Tu|\,|Tv|}$$

$$= \frac{u \cdot v}{|u|\,|v|}$$

$$= \cos\Theta_{u,v}$$





$\Theta_{Tu,Tv} = \Theta_{u,v}$

angle is preserved

Skew-symmetric tensors
Well, they too also represent rotation!

$W = -W^t$

$u \cdot Wu = u_i W_{ij} u_j \qquad L1$

s.sym, sym

$= u_j W_{ji} u_i \qquad W_{ji} = -W_{ij}$

$= u_i (-W_{ij}) u_j = -L1$

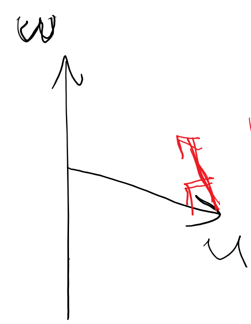
$= -u \cdot W \cdot u$

$\Rightarrow \boxed{u \cdot Wu = 0}$

large def rotation

Wu → skew symmetric

small rotation approx.



$Wu \perp u$

$u \cdot Wu = 0$

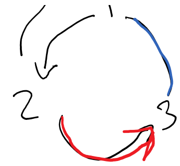
Definition of axis of a second order skew-symmetric tensor

$$W = \begin{bmatrix} 0 & W_{12} & -W_{31} \\ -W_{12} & 0 & W_{23} \end{bmatrix}$$

$W_{ij} = -W_{ji}$

$W_{11} = -W_{11} \Rightarrow W_{11} = 0$

$$W = \begin{bmatrix} 0 & W_{12} & -W_{31} \\ -W_{12} & 0 & W_{23} \\ W_{31} & -W_{23} & 0 \end{bmatrix}$$



$W_{11}' = -W_{11} \Rightarrow W_{11} = 0$

$$Wu = \begin{bmatrix} 0 & W_{12} & -W_{31} \\ -W_{12} & 0 & W_{23} \\ W_{31} & -W_{23} & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} W_{12} u_2 - W_{31} u_3 \\ -W_{12} u_1 + W_{23} u_3 \\ W_{31} u_1 - W_{23} u_2 \end{bmatrix}$$

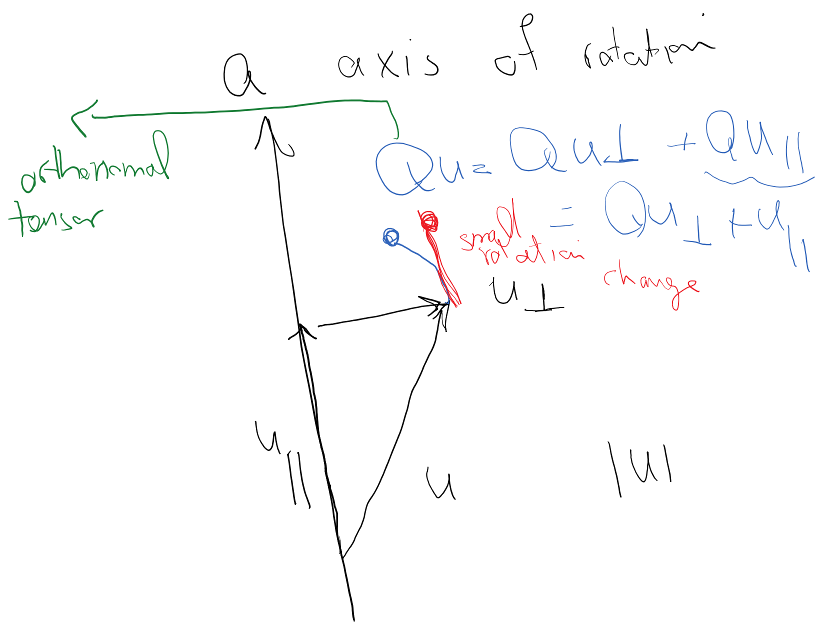
$$= \underbrace{\begin{bmatrix} W_{23} \\ W_{31} \\ W_{12} \end{bmatrix}}_{= \,\text{ax}\, W \text{ or } \omega} \times \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$\boxed{\begin{aligned} \omega &= \text{ax}\, W \\ W &= \text{ax}\, \omega \end{aligned}}$$

1-1 correspondence between vectors & skew sym. 2nd order tensors

$$\boxed{\begin{aligned} \omega_i &= -\frac{1}{2} \epsilon_{ijk} W_{jk} \\ W_{ij} &= -\epsilon_{ijk} \omega_k \end{aligned}}$$

Rotation:

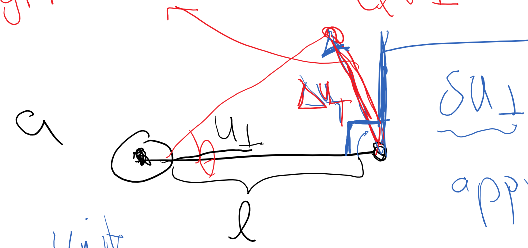


Look at this from the top

Real change of location

$\Delta u_\perp = Qu_\perp - u_\perp$

arc length $= l\theta$

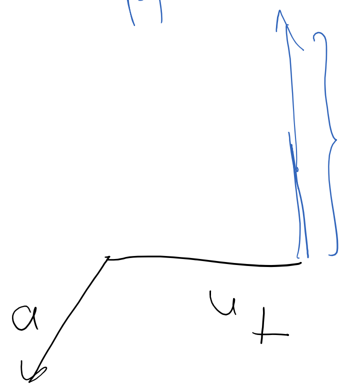


length $= l\theta$ = arc length

$|u_\perp| = l$

unit vector

$\omega = \theta a$



$\omega \times u_\perp =$

magnitude $|\omega|\,|u_\perp| \sin\phi_{\omega, u_\perp}$

$\theta \qquad l \qquad 1$

$\omega u_\perp \approx Qu - u$