2019/09/23

Monday, September 23, 2019 11:23 AM

Recall that any 2nd order tensor can be decomposed to its symmetric and skew-symmetric parts:

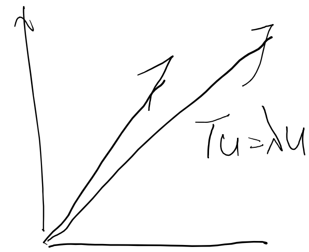
$$T = S + W \qquad S = \operatorname{Sym} T = \frac{T + T^{t}}{2}$$

$$W = \operatorname{Skew} T = \frac{T - T^{t}}{2} \longrightarrow \text{small rotation}$$

Today's focus will be on symmetric 2nd order tensors

Eigenvalues and eigenvectors

$$T u = \underbrace{\lambda}_{\text{eigenvalue}} \overbrace{u}^{\text{eigen vector}}$$



Assume T is n by n and has n distinct and linearly independent eigenvectors?

$$\left.\begin{aligned} T u_{(1)} &= \lambda_1 u_{(1)} \\ &\vdots \\ T u_{(n)} &= \lambda_n u_{(n)} \end{aligned}\right\} \qquad T_{n\times n}\, u_{(i)} = \lambda_{\underline{i}}\, u_{(i)} \quad \text{no summation on } i$$

$$T\left[\, u_{(1)} \,\middle|\, u_{(2)} \,\middle|\, \cdots \; u_{(n)} \right] = \left[\, \lambda_1 u_{(1)} \,\middle|\, \cdots \,\middle|\, \lambda_n u_{(n)} \right] =$$

$$\underbrace{\left[\, u_{(1)} \,\middle|\, \cdots \,\middle|\, u_{(n)} \right]}_{\text{eigenvector matrix}} \underbrace{\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}}_{\Lambda = \operatorname{Diag}(\lambda_i) \text{ eigenvalue matrix}}$$

$$TU = U\Lambda \qquad \times\, U^{-1} \; (u_{(i)}\text{'s are linearly independent})$$

$$\boxed{T = U\Lambda U^{-1}} \quad T \text{ is diagonalizable tensor}$$

$$T^2 = (U\Lambda U^{-1})(U\Lambda U^{-1}) = U\Lambda \underbrace{U^{-1}U}_{I}\Lambda U^{-1}$$

$$= U\Lambda^2 U^{-1} = U\begin{bmatrix} \lambda_1^2 & & \\ & \ddots & \\ & & \lambda_n^2 \end{bmatrix} U^{-1}$$

$$T^{100} = U\Lambda^{100}U^{-1}$$

$$e^T = T + \frac{1}{2!}T^2 + \frac{1}{3!}T^3 + \cdots = U\Lambda U^{-1} + \frac{1}{2}U\Lambda^2U^{-1} + \cdots + \frac{1}{m!}U\Lambda^m U^{-1}$$

$$= U\begin{bmatrix} \underbrace{\lambda_1 + \frac{\lambda_1^2}{2} + \frac{\lambda_1^3}{3!} + \cdots}_{e^{\lambda_1}} & & \\ & \underbrace{\lambda_2 + \frac{\lambda_2^2}{2} + \cdots}_{e^{\lambda_2}} & \\ & & \lambda_n + \frac{\lambda_n^2}{2} + \cdots \end{bmatrix} U^{-1}$$

$$\boxed{e^T = U\begin{bmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{bmatrix} U^{-1} = U e^{\Lambda} U^{-1}}$$

$$\boxed{f(T) = U f(\Lambda) U^{-1} = U\begin{bmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{bmatrix} U^{-1}}$$

Are all 2nd order tensors diagonalizable?

- If a matrix has n distinct eigenvalues their corresponding eigenvectors are linearly independent

$$\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$$

- Matrix may not be diagonalizable if it has repeated eigenvalues

$$T = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad \overbrace{\underbrace{\lambda_{1,2}}_{\text{eigenspace}} = 3}^{\text{algebraic multiplicity } = 2} \qquad \lambda_3 = 5$$

$$\begin{bmatrix} 0 \\ \alpha \\ \beta \end{bmatrix} \longrightarrow \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{linearly independent}}$$

$\underbrace{\text{linearly independent}}$ geometric multiplicity $= 2$

$$T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$\lambda_1 = \lambda_2 = 1 \qquad m_A(1) = 2$

it only has 1 eigenvector

$m_G(1) = 1 \qquad \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\longrightarrow$ In general $T = U J U^{-1}$

$$\begin{bmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_2 & & & \\ & & & \underbrace{\lambda_m \; 1 \; \lambda_m \; \lambda_m}_{\text{repeated}} & & \\ & & & & \ddots & \\ & & & & & \lambda_n \end{bmatrix}$$

Jordan form

Eigen-analysis for symmetric tensors

Example, a symmetric matrix in 3D:

$$A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$$

These properties hold for Hermitian matrices: https://en.wikipedia.org/wiki/Hermitian_matrix

Some properties of symmetric matrices:

1. Eigenvalues are real

$$Au = \lambda u \qquad \lambda \in \mathbb{R}$$

$$\bar{u}\, A u = \lambda\, \underbrace{\bar{u} \cdot u}_{\in \mathbb{R}}$$

complex conjugate

show $\in \mathbb{R}$

2. Eigenvectors are normal for distinct eigenvalues

$$\left.\begin{array}{l} A u_{(1)} = \lambda_1 u_{(1)} \\ A u_{(2)} = \lambda_2 u_{(2)} \end{array} \quad \lambda_1 \neq \lambda_2 \right\} \Longrightarrow \quad u_{(1)} \perp u_{(2)} \quad (u_{(1)} \cdot u_{(2)} = 0)$$

$$\left.\begin{array}{l} u_{(2)} \cdot A u_{(1)} = \lambda_1 u_{(2)} \cdot u_{(1)} \\ u_{(1)} \cdot A u_{(2)} = \lambda_2 u_{(1)} \cdot u_{2} \\ \quad \parallel \\ u_{(2)} A^T u_{(1)} \\ \quad\ A \\ = u_{(2)} \cdot A u_{(1)} \end{array}\right\} \qquad \lambda_1 = u_{(1)} \cdot u_{(2)} = \lambda_2 u_{(1)} \cdot u_{(2)}$$

$$\left.\begin{array}{l} (\lambda_1 - \lambda_2)\ u_{(1)} \cdot u_{(2)} = 0 \\ \lambda_1 \neq \lambda_2 \end{array}\right\} \Longrightarrow$$

$$u_{(1)} \cdot u_{(2)} = 0 \qquad \underline{u_1 \perp u_2}$$

In fact, we can make eigenvectors orthonormal

$$A u_{(1)} = \lambda_1 u_{(1)} \qquad e_1^* = \frac{u_1}{|u_1|} \qquad A e_1^* = \lambda_1 e_1^*$$

$$A u_{(2)} = \lambda_2 u_{(2)} \longrightarrow e_2^* = \frac{u_2}{|u_2|} \qquad A e_2^* = \lambda_2 e_2^*$$

$\vdots$

$$e_i^* \cdot e_j^* = \delta_{ij}$$

so far as long as all eigenvalues are distinct

What happens when eigenvalues are not distinct

For symmetric matrices even if we have repeated eigenvalues the matrix is diagonalizable and we can choose n (dimension of matrix) orthonormal eigenvectors.

Example:

2D

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix} \qquad Au = \lambda u \Rightarrow \det(A - \lambda I) = 0$$

$$\det \begin{bmatrix} a-\lambda & b \\ b & d-\lambda \end{bmatrix} = 0 \Longrightarrow$$

$$\lambda^2 - \underbrace{(a+d)}_{\text{trace} A}\lambda + \underbrace{ad - bc}_{\det A} = 0$$

$$\longrightarrow \quad \lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-b^2)}}{2}$$

$$\lambda = \frac{(a+d) \pm \sqrt{(a-d)^2 + 4b^2}}{2} \qquad = 0$$

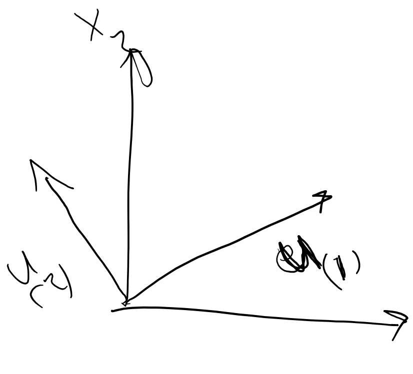
Repeated eigenvalues $\quad a = d$, $b = 0$

$$A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \qquad \lambda_{1,2} = a$$

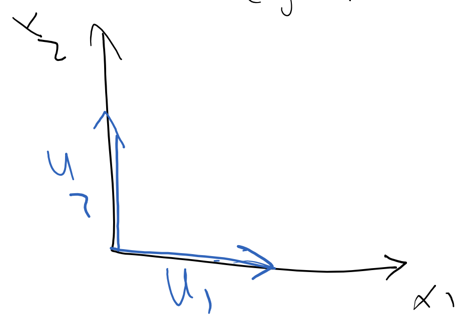
Eigenvectors : Any vector in 2D is an eigenvector!

$A = aI$ $\qquad$ $Au = (aI)u = au$

$\downarrow$ eigenvalue



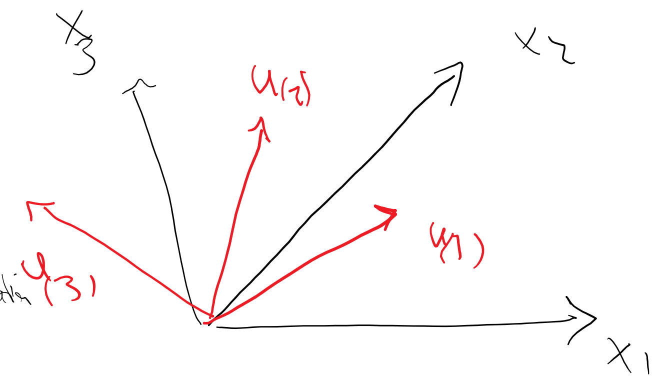
$u_{(1)} \perp u_{(2)}$



3D

$$A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$$

1) $\lambda_1 \neq \lambda_2 \quad \lambda_2 \neq \lambda_3 \quad \lambda_1 \neq \lambda_3$ distinct eigenvalues

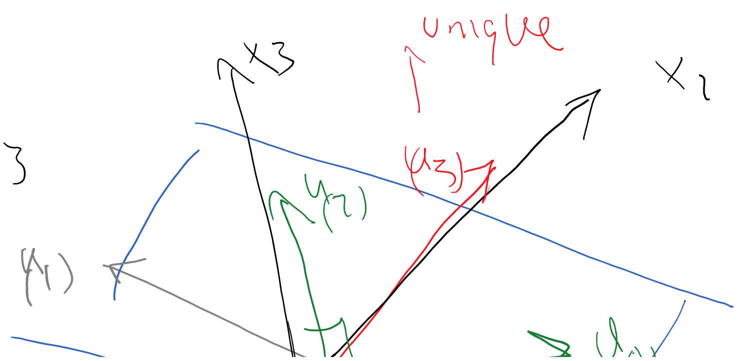


Unique directions for u(1), u(2), u(3) and then having these Normal eigenvectors we can normalize them

$A\, u_{(i)} = \underline{\lambda_i}\, u_{(i)}$ no summation

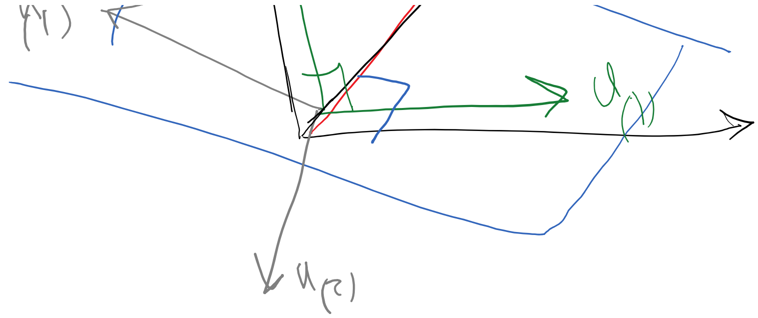
$u_{(i)} \cdot u_{(j)} = \delta_{ij}$

2) $\lambda_1 = \lambda_2 \quad \lambda_2 \neq \lambda_3$



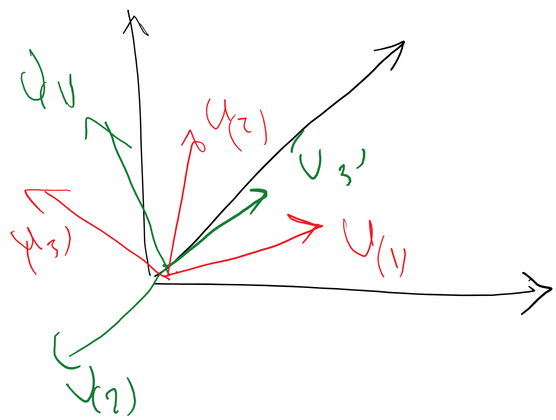
$$T = U\begin{bmatrix} \lambda_1 & & \\ & \lambda_1 & \\ & & \lambda_3 \end{bmatrix}U^{-1}$$

equal



3) $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$

$$T = U^{-1}\underbrace{\begin{bmatrix} \lambda & & \\ & \lambda & \\ & & \lambda \end{bmatrix}}_{\lambda I}U = \begin{bmatrix} \lambda & & \\ & \lambda & \\ & & \lambda \end{bmatrix}$$



How to calculate eigenvalues of a symmetric matrix in 3D?

$$(S - \underset{\text{eigenvalue}}{\sigma} I)\, u = 0 \quad \longrightarrow \quad \det(S - \sigma I) = 0$$

$$\det\begin{bmatrix} S_{11}-\sigma & S_{12} & S_{13} \\ S_{21} & S_{22}-\sigma & S_{23} \\ S_{31} & S_{32} & S_{33}-\sigma \end{bmatrix} = 0 \qquad \underset{\text{symmetric}}{S_{ij} = S_{ji}}$$

Cayley-Hamilton equation:
Or characteristic equation of the tensor:

$$\boxed{\begin{aligned}
&-\sigma^3 + I_1\sigma^2 - I_2\sigma + I_3 = 0\\
&I_1 = tr(S) = S_{11} + S_{22} + S_{33}\\
&I_2 = \frac{1}{2}\left[(tr\, S)^2 - tr(S^2)\right] = \frac{1}{2}\left(S_{ii}S_{jj} - S_{ij}S_{ij}\right)\\
&I_3 = \det S
\end{aligned}}$$

We call I1, I2, I3 the invariants of a symmetric second-order tensor

If we have a function of a second order tensor that is objective (does not depend on coordinate frame) then it can be written in terms of eigenvalues of invariants

$$f(S) = f(\underbrace{S_{11}, S_{12}, S_{13}, S_{22}, S_{23}, S_{33}}_{6 \text{ arguments}})$$

$$f_\sigma(\underbrace{\sigma_1, \sigma_2, \sigma_3}_{3 \text{ arguments}})$$

$$= f_I(\underbrace{I_1, I_2, I_3}_{3 \text{ arguments that are easier to compute}})$$

Related theorem
Cayley-Hamilton theorem
A matric satisfies its characteristic polynomial

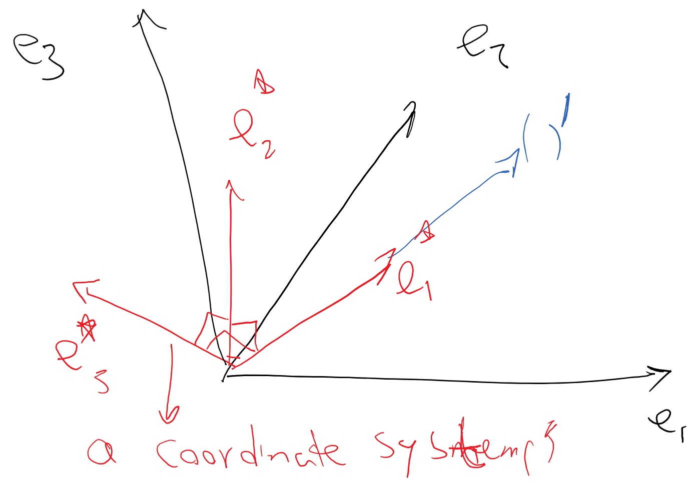
σ^0

$$-\sigma^3 + I_1\sigma^2 - I_2\sigma + I_3\mathbf{1} \quad 0$$

S^0

$$-S^3 + I_1 S^2 - I_2 S + I_3 I = 0$$

Representation of symmetric tensors



$$S u_{(i)} = \sigma_{\underline{i}} u_{(i)}$$

normalize these

$$S e_i^* = \sigma_{\underline{i}} e_i^*$$

$$e_i^* \cdot e_j^* = \delta_{ij}$$

$$S'_{ij} = Q_{im} Q_{jn} S_{mn} \rightarrow S^*_{ij}$$

$$Q = \begin{bmatrix} e_1^* \\ \hline e_2^* \\ \hline e_3^* \end{bmatrix}$$

$$\boxed{S^* = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}}$$

Representation of a symmetric tensor in it's eigenvector coordinate system

$$T_{ij} = e_i \cdot T e_j \qquad \text{Recall}$$

$$S^*_{ij} = e_i^* \cdot \underbrace{(S e_j^*)}_{\sigma_{\underline{j}} e_j^* \text{ no summation}} = e_i^* (\sigma_{\underline{j}} e_j^*) = \sigma_{\underline{j}}\, e_i^* \cdot e_j^*$$

$$\Rightarrow S^*_{ij} = \sigma_j \delta_{ij} \qquad \text{no summation on } j$$

$$\Rightarrow \quad S^*_{ij} = \sigma_{\underline{j}}\, \delta_{ij} \qquad \text{no summation on } j$$

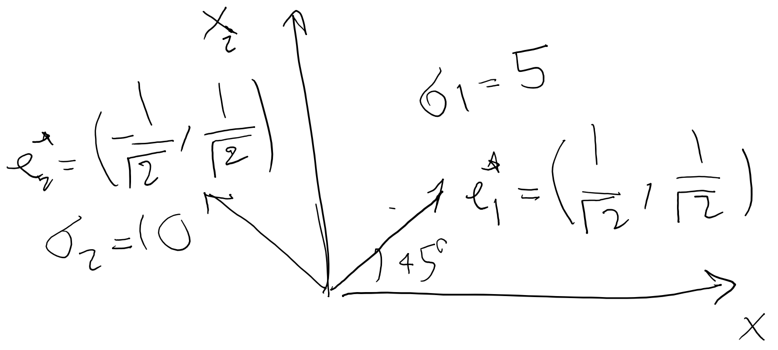
$$[S^*] = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

$$S^* = S^*_{ij}\, e^*_i \otimes e^*_j = \sum_{i=1}^{3} \sigma_i\, e_i \otimes e_i$$

$$\sigma_{\underline{i}}\; e^*_i \otimes e^*_i$$

Example:

The figure shows eigenvectors and eigenvalues of symmetric S:



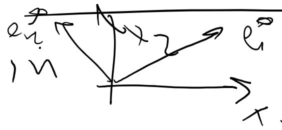
$$[S^*] = 0$$

$$[S] = 0$$

$$[S^*] = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} = \sigma_1\, e^*_1 \otimes e^*_1 + 10\, e^*_2 \otimes e^*_2$$

$\sigma_1 \to 5$, * system, $[1 \;\; 0]$, $[0 \;\; 1]$

$$= 5\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 10\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$



system

$$[S] = \overset{5}{\sigma_1}\, e^*_1 \otimes e^*_1 + \overset{10}{\sigma_2}\, e^*_2 \otimes e^*_2$$

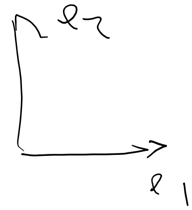
$$e^*_1 = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$$

$$e^*_2 = \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$$

$$= \left[\frac{1}{\sqrt{2}}\right] [\ldots \quad] \; 10 \left[-\frac{1}{\sqrt{2}}\right] [\ldots \quad]$$

$$= 5 \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} + 10 \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ +\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= 5 \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} + 10 \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{15}{2} & -\frac{5}{2} \\ -\frac{5}{2} & \frac{15}{2} \end{bmatrix}$$



Example 2

$$S = \sum \sigma_i \, e_i^* \otimes e_i^*$$

$$S = \begin{bmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & & \sigma_3 \end{bmatrix} \quad \det S = \sigma_1 \sigma_2 \sigma_3$$

$$S^{-1} = \sum \frac{1}{\sigma_i} e_i^* \otimes e_i^*$$

$$\sigma_1, \sigma_2, \sigma_3 \neq 0$$

$$\begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{bmatrix}^* \begin{bmatrix} 1/\sigma_1 & & \\ & 1/\sigma_2 & \\ & & \frac{1}{\sigma_3} \end{bmatrix} = \mathbb{1}$$

$$\ln S = \sum_{i=1}^{3} \ln \sigma_i \, e_i^* \otimes e_i^*$$

Positive definite tensors

$$\text{sym}\, T = \frac{T + T^T}{2} \qquad \text{skew}\, T = \frac{T - T^T}{2}$$

$$u \cdot T u = u \cdot (\text{Sym}\,T + \text{Skew}\,T)\, u$$

$$= u \cdot (\text{Sym}\,T) u + u (\text{Skew}\,T) u$$

$$= u_i u_j (\text{Sym}\,T)_{ij} + \cancel{\underbrace{u_i u_j}_{\text{sym on } i,j} \underbrace{(\text{Skew}\,T)_{ij}}_{\text{skew sym}}} \quad 0$$

$$\Rightarrow$$

$$\boxed{u \cdot Tu = u \cdot \underline{\text{Sym}\,T}u} \quad \star$$

Skew-symmetric part does not contribute

Positive definite tensor T satisfies the following property:

$$\forall u \quad u \cdot Tu \geq 0 \quad \& \quad u \cdot Tu = 0 \iff u = 0$$

A <u>positive</u> or semi-positive definite tensor satisfies

$$\forall u \quad u \cdot Tu \geq 0$$

Because of (*) we only need to work with symmetric tensors (i.e.
symmetric part of a tensor) when talking about positive definiteness

$$T = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{not p. def}$$

$$\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \quad \& \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq 0$$

$$T = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \quad uTu = u_1^2(5) + u_2^2(3) \geq 0$$

$$= 0 \text{ iff } \bar{u} = 0$$

$$T = \begin{bmatrix} 5 & 0 \\ 0 & -3 \end{bmatrix} \quad u \cdot Tu = 5u_1^2 - 3u_2^2 \not\geq 0$$

$$T = \begin{bmatrix} & \\ 0 & -3 \end{bmatrix} \qquad u \cdot Tu = 5u_1^2 - 5u_2 \not\geq 0$$

$$u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

side note

$\cancel{T}$ pos def $\qquad \|u\|_T = \sqrt{u \cdot Tu}$

this is a norm