2019/09/25

Wednesday, September 25, 2019 11:32 AM

Summary from last time on positivity of symmetric matrices

$$S^* = \begin{bmatrix} s_1 & & & \\ & s_2 & & \\ & & \ddots & \\ & & & s_n \end{bmatrix} \quad \text{in its principal axes}$$

$$u^* \cdot S^* \cdot u^* = \sum_{i=1}^{n} u_i s_i^* \geq 0 \quad \text{iff } s_i \geq 0$$

$$> 0 \quad \text{iff } s_i > 0$$

An example of how to form positive 2nd order tensors:

$$F \implies C = F^t F \geq 0$$

$$> 0 \quad \text{iff } \det F \neq 0$$

$$u \cdot (F^t F u) \geq 0$$

$$\underbrace{Fu}_{v} \cdot \underbrace{Fu}_{v} \geq 0$$

$$> 0 \quad \text{if } \det F \neq 0 \qquad v \cdot v = 0 \Rightarrow \underset{\det F \neq 0}{Fu = 0} \implies u = 0$$

Square root of a positive 2nd order tensor

Theorem 109:

S in Psym with principal frame $\{x^*, e_i^*\}$ such that

$$S = \sigma_1 e_1^* \otimes e_1^* + \sigma_2 e_2^* \otimes e_2^* + \sigma_3 e_3^* \otimes e_3^* \qquad [S^*] = \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{bmatrix}$$

$$\sigma_i > 0$$

there is a unique $U \in Psym$ such that $U^2 = S$ $\qquad U = \sqrt{S}$

$$\boxed{U = \begin{bmatrix} +\sqrt{\sigma_1} & & \\ & +\sqrt{\sigma_2} & \\ & & +\sqrt{\sigma_3} \end{bmatrix}}$$

$\sqrt{9} = \cancel{\pm} 3$ looking for >0 root

Theorem 112 (Polar Decomposition Theorem) *Let* $\mathbf{F} \in \mathrm{Inv}\,\mathcal{V}$. *Then* $\exists$ *a unique pair of tensors* $\mathbf{U}, \mathbf{V} \in \mathrm{Psym}$ *and a unique* $\mathbf{R} \in \mathrm{Orth}\,\mathcal{V} \ni$

positive def tensors

Rotation

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}.$$

(Lagrangian) Solid

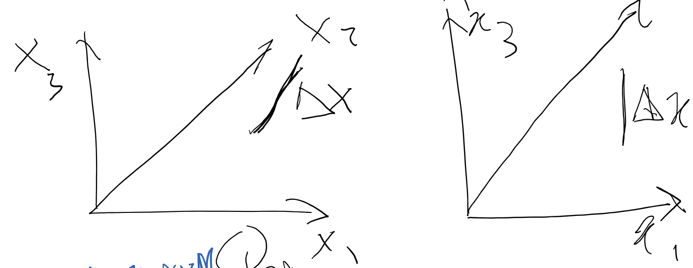
Moreover, $\det \mathbf{R} = +1$ *or* $\det \mathbf{R} = -1$, *depending as* $\det \mathbf{F} > 0$ *or* < 0.

Use of this

x deformed configuration

X undeformed

$F = \nabla_X x$



$$\underbrace{\Delta x}_{} = \underbrace{F \Delta X}_{}$$

$\underbrace{R}\,\underbrace{U}$

2nd rotate it 1st stretch it

$V R$

2nd stretch 1st rotate

Large deformation: decomposition is multiplicative
Small deformation: decomposition is additive

$R = R_3 R_2 R_1$

small $W = W_1 + W_2 + W_3$

change of location by rotation.

large deformation

$F = RU = VR$

small deformation:

$$H = F - I = \underbrace{\tfrac{1}{2}(H + H^T)}_{E} + \underbrace{\tfrac{1}{2}(H - H^T)}_{}$$

$= \nabla u_X$

strain Skew rotation

Motivation for proof

$F = RU$

$$\underbrace{F^t F}_{\Downarrow\ \mathrm{Psym}\ (>0)} = (RU)^t (RU) = \underbrace{U^t}_{\downarrow\ U} \underbrace{(R^t R)}_{I} U = U I U = U^2$$

① $\boxed{U = \sqrt{F^t F}}$ Idea what U is

Obviously $U \in$ PSym

R = rotation?

$F = RU \Rightarrow R = FU^{-1}$

$R^t R = ?$ $(FU^{-1})^t F U^{-1} = U^{-t} \underbrace{(F^t F)}_{U^2} U^{-1} = U^{-1}(U^2)U^{-1} = I$

sym

we proved
$$\boxed{\begin{array}{l} F = RU \\ U = \sqrt{F^t F} \\ R \text{ orthogonal} \end{array}}$$

appropriate for solids

Second part $F = VR$

$V = FR^{-1} \longrightarrow \underline{V = \sqrt{FF^t}}$
we can show this

$F = RU = VR \Longrightarrow \boxed{V = RUR^t}$

V is sym? $x \cdot RUR^t y = R^t x \cdot UR^t y = \underset{U \in Sym}{\underset{\sim}{U^t}} R^t x \cdot R^t y$

$= RUR^t x \cdot y = y \cdot \underbrace{RUR^t}_{V} x$

$x \cdot Vy = y \cdot Vx \Longrightarrow$ V is sym ✓

$x \cdot Vx > 0$
positive

$x \cdot Vx = x \cdot RUR^t x = \underbrace{R^t x}_{y} \cdot U (\underbrace{R^t x}_{y})$

$= y \cdot Uy > 0$
U is PSym

we have shown $V \in$ PSym

In addition we want to show $V^2 = FF^T$

$V = RUR^t \quad V^2 = RU\underbrace{R^tR}_{I}UR^t = R\underbrace{U^2}_{F^tF}R^t$

$= R\underbrace{F^tF}_{U^2}R^t$

$F = RU \Rightarrow R = FU^{-1}$ $\Bigg\} \Rightarrow FU^{-1}(U^2)\underbrace{U^{-t}}_{U^{-1}}F^t =$

$\boxed{V = FF^t}$

$F = RU \Longrightarrow \underline{\det F = \det R \underbrace{\det U}_{>0}}$ ← PSym

$\det R = \begin{cases} +1 & \text{rotation} \longrightarrow \det F > 0 \\ -1 & \text{" + reflection} \quad \det F < 0 \end{cases}$

In practice (next section) det F > 0 so a deformation is always decomposed to a pure stretch and a rotation.

$F = \underset{\text{rotation}}{R}U = VR \qquad U = \sqrt{F^tF}$ Lagrangian

$V = \sqrt{FF^t}$ Eulerian

R rotation $\det F > 0$

" + reflection $\det F < 0$

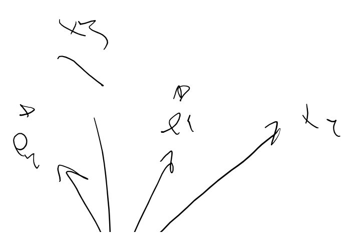
Theorem 106 *Let*

$$f(S_{ij}) := f(S_{11}, S_{22}, S_{33}, S_{12}, S_{13}, S_{23})$$

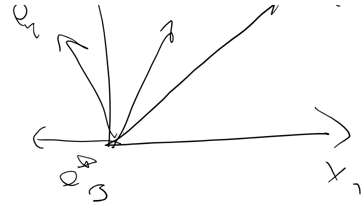
be a scalar invariant of $\mathbf{S} \in \mathrm{Sym}$ *(that is* $f(S_{ij}) = f(S'_{ij})$, *where* S_{ij} *and* S'_{ij} *are components of* $\mathbf{S}$ *w.r.t. two frames* X *and* X'*). Then* $\exists$ *a unique real-valued function* g *of three real variables* $\ni$

$$\boxed{f(S_{ij}) = g(I_1(\mathbf{S}), I_2(\mathbf{S}), I_3(\mathbf{S}))}$$

where $I_i(\mathbf{S})$ *are the fundamental invariants of* $\mathbf{S}$*. Conversely, any function of this form is a scalar invariant of* $\mathbf{S}$*.*



of this form is a scalar invariant of S.



$S \in Sym$ express it in principal directions

$$[S^*] = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

$$f(S) = f(\sigma_1, \sigma_2, \sigma_3, 0, 0, 0)$$
$$= f_\sigma(\sigma_1, \sigma_2, \sigma_3)$$

$$-\sigma^3 + I_1\sigma^2 + I_2\sigma - I_3 = 0$$

$$\sigma_1, \sigma_2, \sigma_3 \Longleftrightarrow I_1, I_2, I_3$$

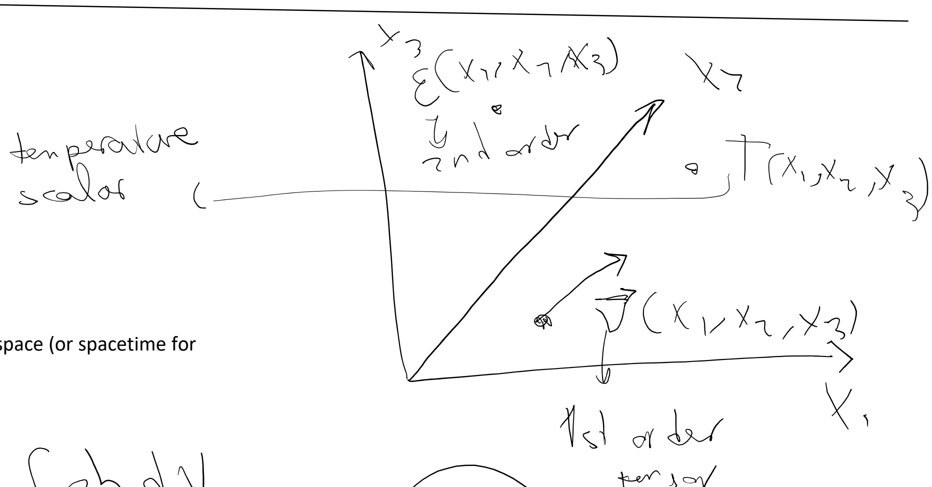
$\sigma_1+\sigma_2+\sigma_3$ $\qquad$ $\sigma_1\sigma_2\sigma_3$

easier

$$f(S) = f_I\left(I_1(\sigma_1,\sigma_2,\sigma_3), I_2(\sigma_1,\sigma_2,\sigma_3), I_3(\sigma_1,\sigma_2,\sigma_3)\right)$$

Calibrate your model based on I_1, I_2, I_3

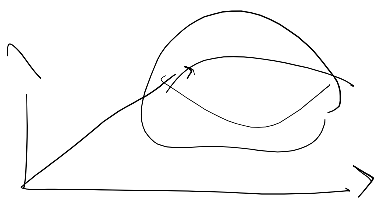
Tensor fields



These are tensor functions that depend on space (or spacetime for dynamic problems) coordinates

$$\int_{\partial D} \sigma \cdot n \, dS = \int_D \rho b \, dV$$

$$\int_D \operatorname{div} \sigma \, dV = \int_D \rho b \, dV$$



$$\Rightarrow \quad \operatorname{div} \sigma - \rho b = 0$$

What is the divergence of a tensor

magnetic field

$q = -k\nabla T.$

$\underbrace{q = -k \nabla T}$

magnetic field

$\dot{D} - \underbrace{\nabla \times H}_{\text{curl}} + \underbrace{J}_{\text{electric current}} = 0$ from Maxwell's eqns

electric flux

T is m's order tensor field

$T = T_{i_1 \ldots i_m} e_{i_1} \otimes \cdots e_{i_m}$

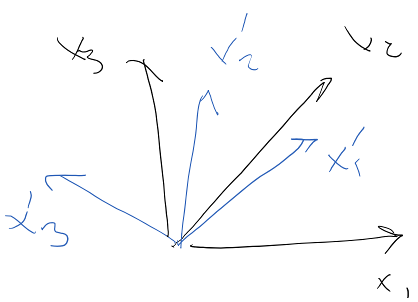
$\underbrace{\nabla T}_{\text{grad of } T} = T_{i_1 \ldots i_m, j}\, e_{i_1} \otimes \cdots \otimes e_{i_m} \otimes e_j$

$T_{i_1 \ldots i_m, j} = \dfrac{\partial T_{i_1 \ldots i_m}(x_1, x_2, x_3)}{\partial x_j}$

Since the definition is based on coordinate components, we need to show that it's coordinate-independent

Let's show this for a second order tensor

$T = T_{ij}\, e_i \otimes e_j$



given

$(\nabla T)_{ijk} = T_{ij,k}\, e_i \otimes e_j \otimes e_k$

$(\nabla T)'_{mnp} = \dfrac{\partial T'_{mn}(x_1', x_2', x_3')}{\partial x_p'}$

$\to (\nabla T)'_{mnp} = Q_{mi} Q_{nj} Q_{pk} (\nabla T)_{ijk}$

show

constant

$\nabla T'_{mnp} = \dfrac{\partial T'_{mn}}{\partial x_p'} = \dfrac{\partial \left(\overbrace{Q_{mi} Q_{nj} T_{ij}}^{T'_{mn}} \right)}{\partial x_p'} = Q_{mi} Q_{nj} \dfrac{\partial T_{ij}(x_1, x_2, x_3)}{\partial x_p'}$

3 p's X

$$= Q_{mi} Q_{nj} \underbrace{\frac{\partial T_{ij}}{\partial x_k} \frac{\partial x_k}{\partial x'_p}}_{\text{chain rule}}$$

$$x_k = Q_{pk} x'_p$$
$$x_k = Q_{rk} x'_r$$

$$= Q_{mi} Q_{nj} \frac{\partial Q_{rk} x'_r}{\partial x'_p} \frac{\partial T_{ij}}{\partial x_k} =$$

$$Q_{mi} Q_{nj} Q_{rk} \underbrace{\left(\frac{\partial x'_r}{\partial x'_p}\right)}_{\delta_{rp}} \frac{\partial T_{ij}}{\partial x_k}$$

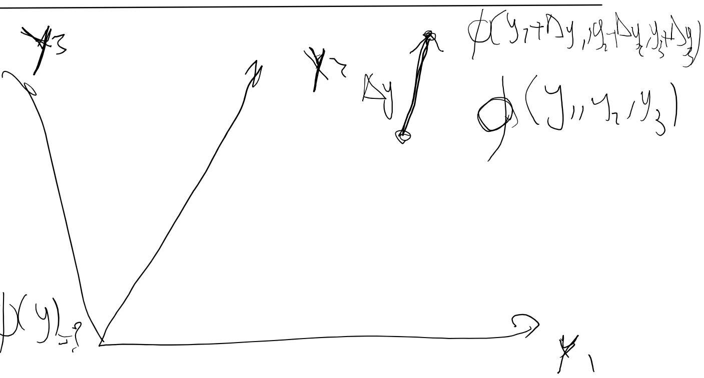
$$= Q_{mi} Q_{nj} Q_{rk} \delta_{rp} \, (\nabla T)_{ijk}$$

$$\Longrightarrow \boxed{(\nabla T')_{mnp} = Q_{mi} Q_{nj} Q_{pk} (\nabla T)_{ijk}}$$

So ∇T is a 3rd order tensor :)

Interpretation of gradient:

$$\underbrace{\Delta y}_{\text{change of location}} = (\Delta y_1, \Delta y_2, \Delta y_3)$$



we want $\Delta \phi = \phi(y + \Delta y) - \phi(y)$

Taylor's expansion

$$\Delta \phi = \phi(y_1 + \Delta y_1, y_2 + \Delta y_2, y_3 + \Delta y_3) - \phi(y_1, y_2, y_3)$$

$$= \phi(y_1, y_2, y_3) + \underbrace{\left(\frac{\partial \phi}{\partial y_1} \Delta y_1 + \frac{\partial \phi}{\partial y_2} \Delta y_2 + \frac{\partial \phi}{\partial y_3} \Delta y_3\right)}_{\delta \phi}$$

$$+ \left(\frac{1}{2!} \frac{\partial^2 \phi}{\partial y_i \partial y_j} \Delta y_i \Delta y_j\right) + \cdots - \phi(y_1, y_2, y_3)$$

$\delta\phi$

$$\Delta\phi = \overbrace{\underbrace{\begin{bmatrix}\phi_{,1} & \phi_{,2} & \phi_{,3}\end{bmatrix}}}^{\text{covector}} \begin{bmatrix}\Delta y_1 \\ \Delta y_2 \\ \Delta y_3\end{bmatrix} + \frac{1}{2}\begin{bmatrix}\Delta y_1 & \Delta y_2 & \Delta y_3\end{bmatrix} \underbrace{\begin{bmatrix}\phi_{,11} & \phi_{,12} & \phi_{,13} \\ & \ddots & \\ \phi_{,31} & & \phi_{,33}\end{bmatrix}}_{\text{Hessian}} \begin{bmatrix}\Delta y_1 \\ \Delta y_2 \\ \Delta y_3\end{bmatrix} + HOT$$

$\phi:$ $\nabla\phi = \phi_{,j}\, e_j$ gradient

$$\boxed{\Delta\phi = \text{grad}\,\phi\;\Delta y}$$

For any tensor $T(y)$

$$\boxed{\underset{m\text{th order}}{\Delta T} \approx \underbrace{(\text{grad}\,T)}_{m+1\text{th order}}\;\underbrace{\Delta y}_{\text{vector}}}$$

Vector field

$V = V_1 e_1 + V_2 e_2 + V_3 e_3$

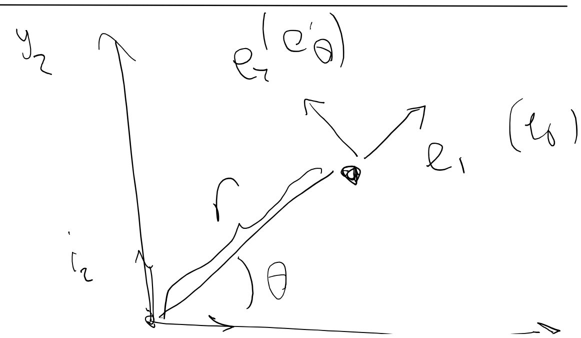
$$\boxed{\text{grad}\,V = \begin{bmatrix} V_{1,1} & V_{1,2} & V_{1,3} \\ V_{2,1} & V_{2,2} & V_{2,3} \\ V_{3,1} & V_{3,2} & V_{3,3} \end{bmatrix}}$$

$(\text{grad}\,V)\,\Delta y = \Delta\vec{V}$

Curvilinear coordinate systems

$y_1 = y_1(x_1, x_2)$

$y_2 = y_2(x_1, x_2)$



Example

$y_1 = \underset{x_1}{r}\cos\underset{x_2}{\theta} = x_1 \cos x_2$

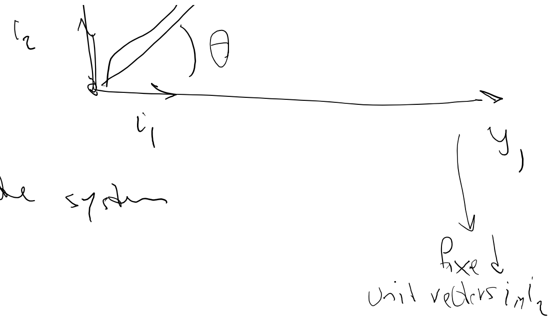
Polar coordinate. $y_2 = r\sin\theta = x_1 \sin x_2$

Example

Polar coordinate system

$y_1 = r\cos\theta = x_1 \cos x_2$

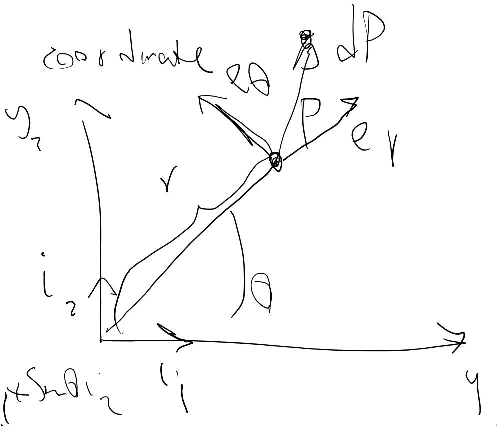
$y_2 = r\sin\theta = x_1 \sin x_2$



x_1, x_2 are curvilinear coordinate system

How to define e_r, e_θ

Brute-force calculation of grad in polar coordinate



$$P = r e_r$$

$$d\vec{P} = d(r e_r) = (dr)e_r + r(de_r)$$

$$e_r = \cos\theta\, i_1 + \sin\theta\, i_2$$

$$de_r = \underbrace{(-\sin\theta\, i_1 + \cos\theta\, i_2)}_{e_\theta} d\theta$$

$$\boxed{dP = \underbrace{(dr)}\, e_r + \underbrace{(r\, d\theta)}\, e_\theta}$$

?

Also $\boxed{\begin{aligned} de_r &= e_\theta\, d\theta \\ de_\theta &= -e_r\, d\theta \end{aligned}}$

$$d\phi(r,\theta) = \frac{\partial \phi}{\partial r} dr + \frac{\partial \phi}{\partial \theta} d\theta$$

$$\boxed{\begin{aligned} d\phi(r,\theta) &= \frac{\partial \phi}{\partial r} dr + \frac{1}{r}\frac{\partial \phi}{\partial \theta}(r\, d\theta) \\ dP &= \underbrace{dP_r}_{dr}\, e_r + \underbrace{dP_\theta}_{r\,d\theta}\, e_\theta \end{aligned}}$$

$$d\phi = \begin{bmatrix} (\nabla\phi)_r & (\nabla\phi)_\theta \end{bmatrix} \begin{bmatrix} dP_r \\ dP_\theta \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial \phi}{\partial r} & \frac{1}{r}\frac{\partial \phi}{\partial \theta} \end{bmatrix} \begin{bmatrix} dr \\ r\,d\theta \end{bmatrix}$$

$$\Longrightarrow \boxed{\nabla\phi = \begin{bmatrix} \frac{\partial \phi}{\partial r} & \underbrace{\frac{1}{r}\frac{\partial \phi}{\partial \theta}} \end{bmatrix}}$$

general

general

$$\nabla \phi = \sum \underbrace{\frac{1}{h_i}}_{\text{scale factors}} \frac{\partial \phi}{\partial x_i} e_i$$

$h_r = 1$
$h_\theta = r$

for polar coordinate