2019/09/30

Monday, September 30, 2019 10:14 AM

Section 2:

# Kinematics:

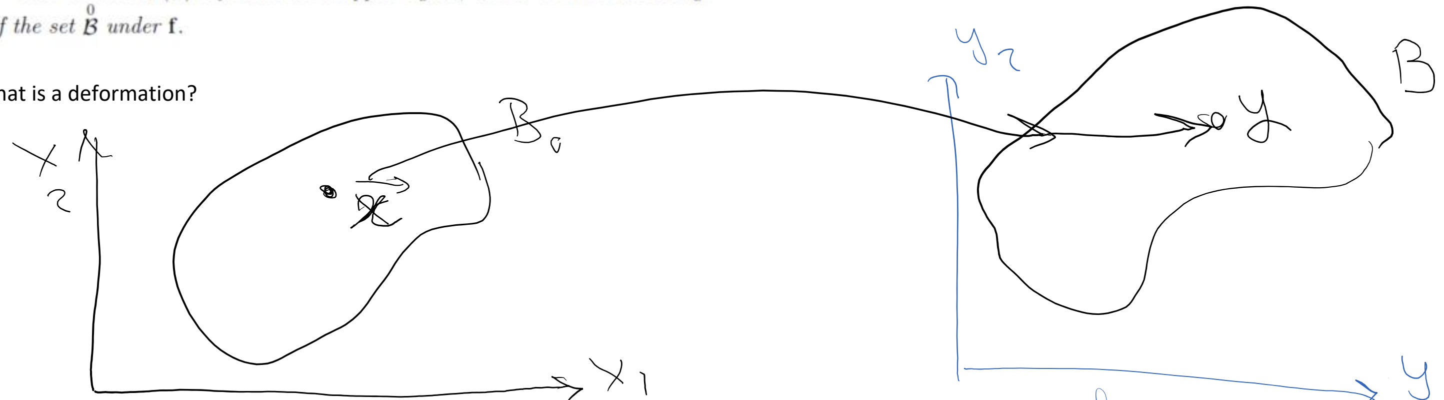
**Definition 72** *Let* $\overset{0}{\mathcal{B}}$ *be an open, bounded, regular region of a Euclidean point space* $\mathcal{E}$. *A* deformation $\mathbf{f}$ *is a mapping (function) of points in* $\overset{0}{\mathcal{B}}$ *onto another open region of* $\mathcal{E}$ *with the properties*

1. $\mathbf{f}$ *is one-to-one; i.e.,* $\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{y}) \Rightarrow \mathbf{x} = \mathbf{y}\ \forall\ \mathbf{x}, \mathbf{y} \in \overset{0}{\mathcal{B}}$,
2. ~~$\mathbf{f} \in C^2(\overset{0}{\mathcal{B}})$, $\mathbf{f}^{-1} \in C^2(\mathbf{f}(\overset{0}{\mathcal{B}}))$,~~
3. $\det \nabla \mathbf{f}(\mathbf{x}) > 0\ \forall\ \mathbf{x} \in \overset{0}{\mathcal{B}}$.

*The notation* $\mathbf{f}(\overset{0}{\mathcal{B}})$ *refers to the mapped region, which is called the* image *of the set* $\overset{0}{\mathcal{B}}$ *under* $\mathbf{f}$.

$y = f(x)$

What is a deformation?



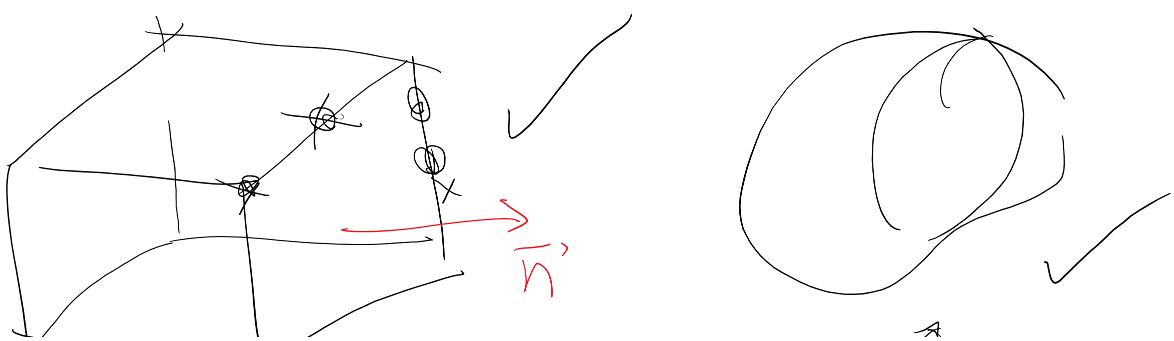

Reference configuration (solid ME)
"Undeformed" = 

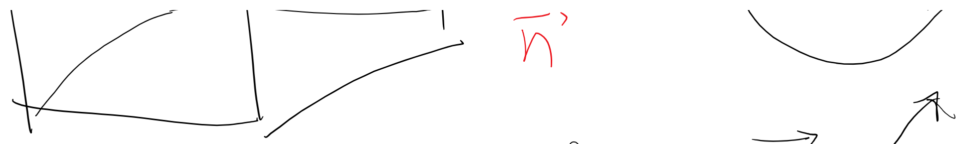
deformed configuration
current =

Other notations $X$ $x$

we use $x$ $y$

Regular : "nice boundaries" that normal vector can be defined almost everywhere (a.e.)






$\vec{n}$ is needed because of $\oint_{net} = \vec{g} \cdot n$

net flux of a quantity

flux vector

Divergence theorem that needs $\vec{n}$ being defined a.e.

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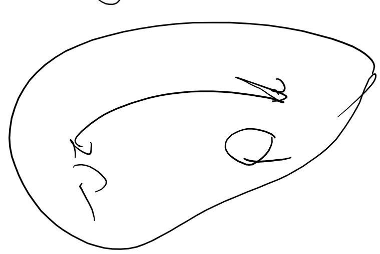
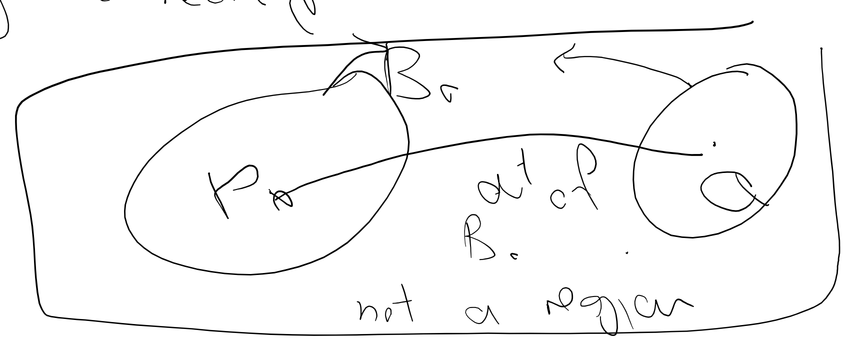
Region : simply connected

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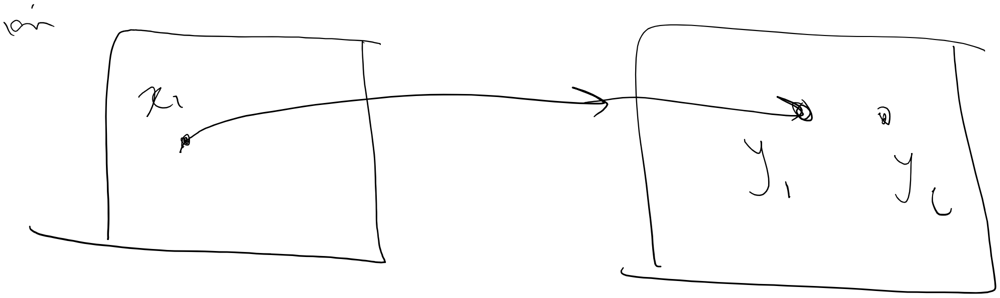
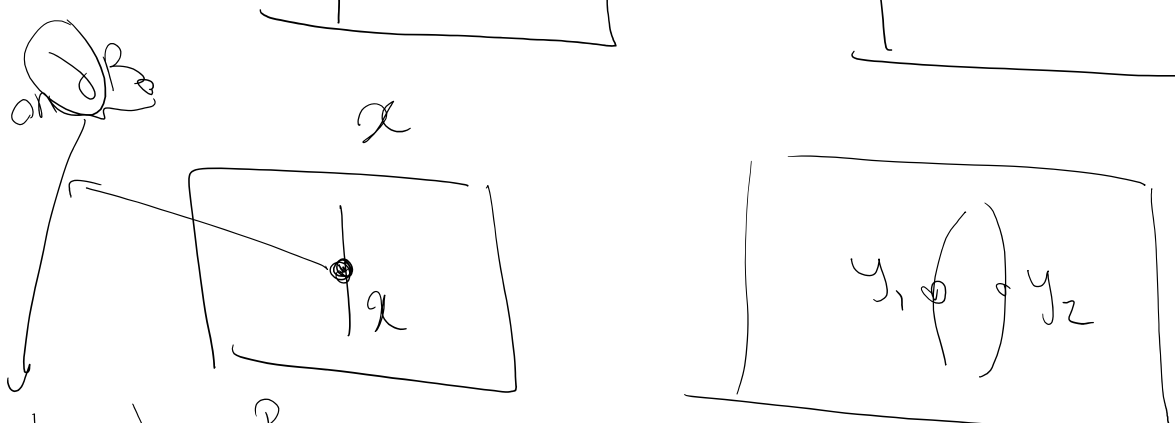
Other properties of f:

1. f is 1-1 function

function



boundary of $B_0$
& we only consider "inside $B_0$" ($B_0$ open)
not a concern

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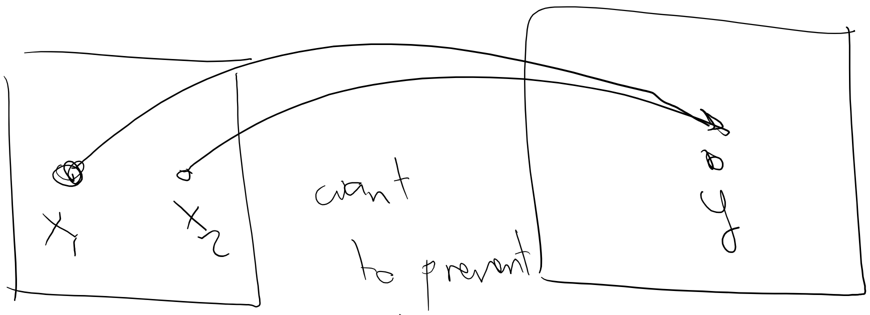
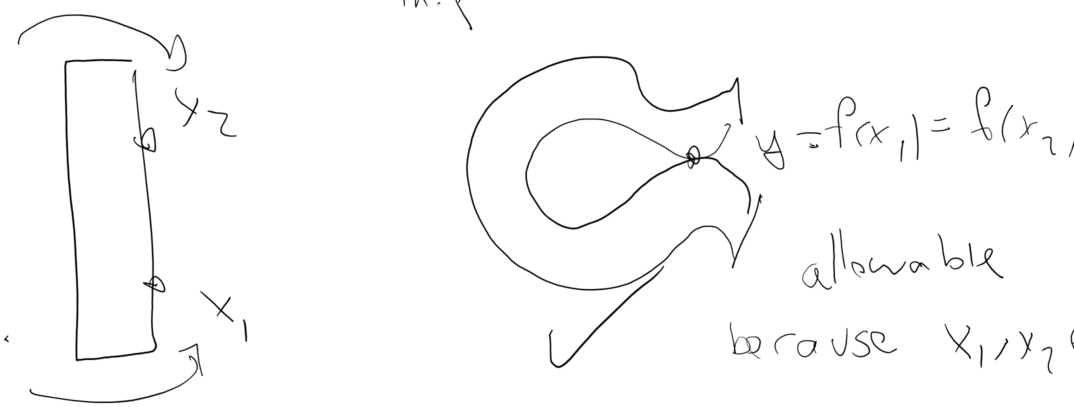
## 1 − 1




want to prevent this




allowable because $x_1, x_2 \in \partial B_0$

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## Property 2

$f$ should be $C^2$ — have two derivatives & all should be continuous

$y$
$y'$
$y''$

1D all exist & are continuous

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1D $\quad u = y - x \quad$ strain $\varepsilon = u_{,x} = y_{,x} - 1$

EOM $\quad \rho \ddot{u} = \sigma_{,x} + \rho b$

$\sigma = E\varepsilon$

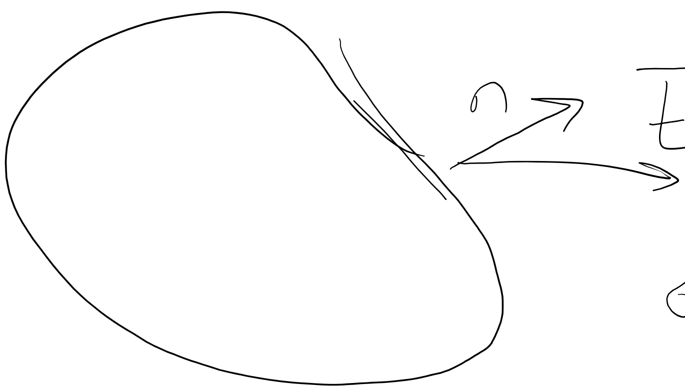
$\longrightarrow$

$\sigma = E\,\varepsilon$

$\downarrow$ Elastic modulus

$$\underbrace{\left(E(y_{,x}-1)\right)_{,x}}_{\text{2 derivatives on } x} + \rho b = \rho\, \underbrace{\ddot{u}}_{}$$

2 derivatives in time if it's a dynamic problem



$\vec{t} = \underset{\sim}{\sigma} \cdot \vec{n}$

stress tensor

$\sigma = \mathbb{C}\,\varepsilon = \mathbb{C}(\underset{\sim}{\nabla y} - I)$

we just need 1 derivative on the boundary of domain to evaluate traction & 2 inside

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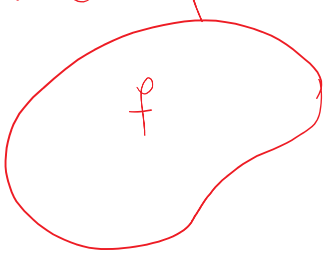
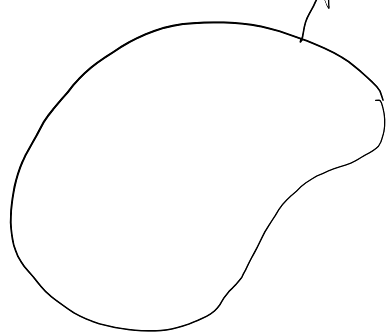
Read Def 73, 74, 75

Trace operator




f is defined in $B_0$ open

$\gamma f = f|_{\partial B_0}$



extend the def to $\partial B$ is possible f is defined in $\overset{\circ}{B}$ open

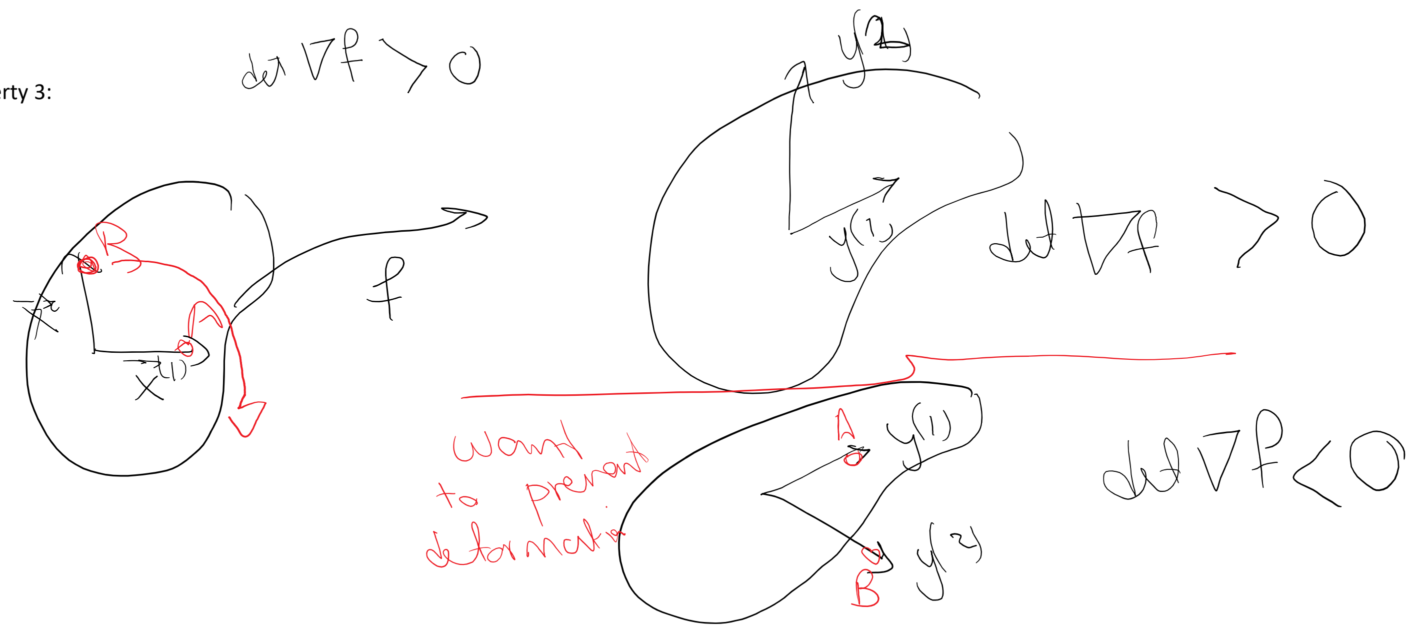
if $f \in C^m(\overset{\circ}{B})$ $\quad \partial f \in C^{m-1}(\overset{\circ}{B})$

$u = f - x$

in PDE $\quad C^2 \; u_{,ij} \longrightarrow u_{,i} = y_{,i} - 1$ needed for traction evaluation

**Remark 28** *The requirement of well-behaved first partial derivatives supports the unambiguous extension of $f$ to the boundary $\partial\mathcal{B}$. Inductively, the trace operator "evaluates" a function $f \in C^M(\overset{0}{\mathcal{B}})$ and its partial derivatives up to order $M-1$ on $\partial\mathcal{B}$. Specifically, for any deformation $\mathbf{f} \in C^2(\overset{0}{\mathcal{B}})$, the trace allows us to "evaluate" the components $f_i$ and the partial derivatives $f_{i,j}$ on $\partial\mathcal{B}$. This is sufficient for a complete kinematic description of the closed body $\mathcal{B}$. These arguments are associated with the following Extension Theorem.*

Property 3: $\quad \det \nabla f > 0$



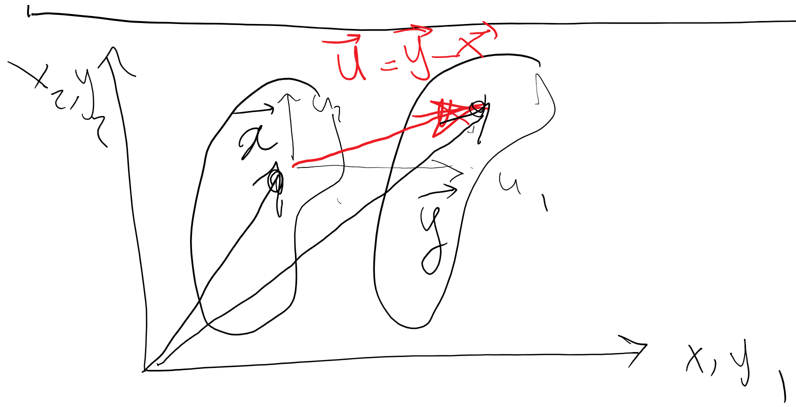

- Next we are going to cover have line segments (length and angle between them), surfaces, and volumes change because of deformation
- Rigid motion

Definition of displacement:

$$\boxed{u(x) = \underbrace{y(x)}_{\text{new position}} - \underbrace{x}_{\text{original position}}}$$

$U$ is 1-1

$U$ is $C^?(B_0)$




$$u_i(x_1, x_2, x_3) = y_i(x_1, x_2, x_3) - x_i \qquad i \in \{1,2,3\}$$

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial y_i}{\partial x_j} - \underbrace{\frac{\partial x_i}{\partial x_j}}_{\delta_{ij}}$$

$$\left(\nabla u_{/x}\right)_{ij} = \left(\nabla y_{/x}\right)_{ij}$$

$$\boxed{\begin{aligned} &\underbrace{\nabla u_{/x}}_{H} = \underbrace{\nabla y_{/x}}_{F} - I \\ &F_{ij} = \frac{\partial y_i}{\partial x_j} \quad \text{gradient of deformation} \\ &H_{ij} = \frac{\partial u_i}{\partial x_j} \quad \text{gradient of displacement} \\ &\underbrace{H}_{\text{small deformation theory}} = F - I \end{aligned}}$$

more useful

finite deformation

Rigid body motion:

Definition 78: a deformation f is rigid iff it preserves distance between all pairs of points

$$|y_1 - y_2| = |x_1 - x_2| \quad \forall\ x_1, x_2 \in B^0$$

Rigid motions are composed of translations and rotations.

Why?

Theorem 119: Let f be a rigid deformation and O the origin. WE define the relative displacement w.r.t. ) as

$$\xi(x) = f(x) - \underbrace{f(0)}_{\text{corresponds to translation}}$$

we claim

① $|\xi(x)| = |x|$

② $\xi(x) \cdot \xi(y) = x \cdot y$

$\xi$ preserves ~~distance~~ magnitude & $\underbrace{\text{angle}}_{\text{inner product}}$

$$\Longrightarrow \xi(x) = \underbrace{Q}_{\text{proper orthogonal tensor (rotation)}} x$$

$$|f(x) - f(0)| = |x - 0|$$

$$|\xi(x)| = |x|$$

$$\xi(x+y) \cdot \xi(x+y) = \xi^2(x) + 2\xi(x)\cdot\xi(y) + \xi^2(y)$$

Use this to show

$$\xi(x)\cdot\xi(y) = x \cdot y$$

And we need to show it's a linear operator (skip)
From theorem 78 for a linear 2nd order tensor that preserves magnitude, distance, or angles -> was an orthogonal tensor

$$y(x) = f(0) + \underbrace{Q x}_{f(x)}$$

or