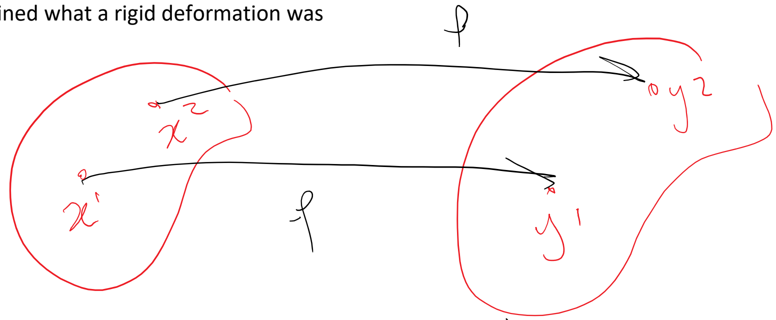
2019/10/02

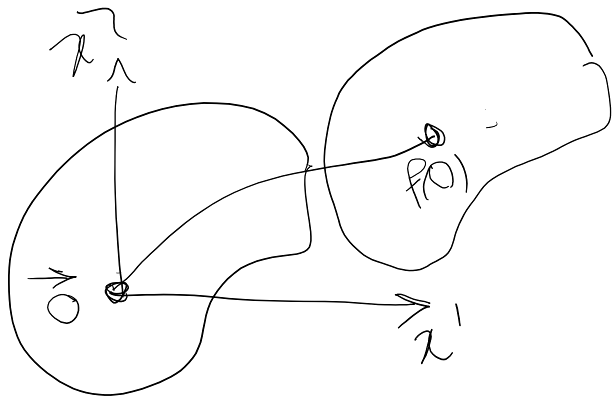
Wednesday, October 2, 2019 11:31 AM

From last time we defined what a rigid deformation was



$$|x^2 - x^1| = |y^2 - y^1|$$

$\Longrightarrow$



$$\left.\begin{aligned} \xi(x) &= f(x) - f(0) \\ |\xi(x)| &= |x| \\ \xi(x)\cdot\xi(y) &= x\cdot y \end{aligned}\right\} \Longrightarrow$$

by showing ξ is linear

$$\xi(x) = \underset{\substack{\downarrow \\ \text{orthogonal tensor}}}{Q}x$$

$$f(x) = f(0) + \xi(x) = f(0) + Qx$$

if this holds $|x^1 - x^2| = |y^1 - y^2|$

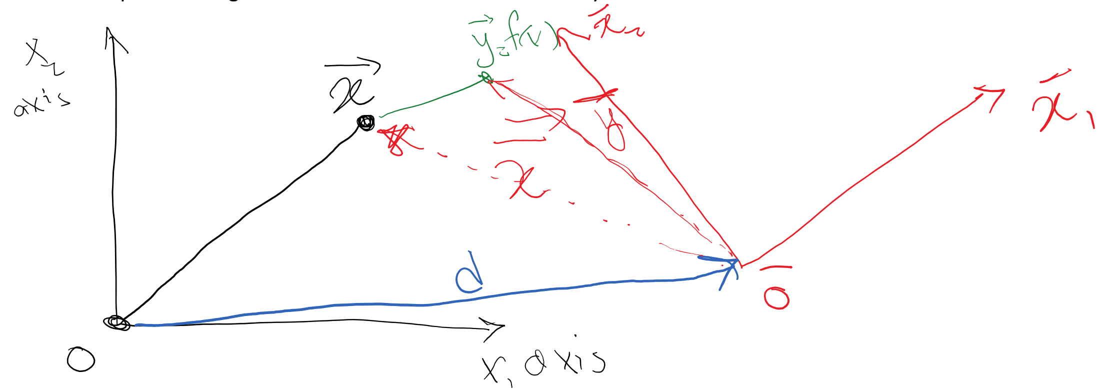
$$\boxed{\begin{array}{ll} \text{Rigid motion} \Longleftrightarrow & \\ f(x) = \overset{\substack{\text{translation} \\ \uparrow}}{c} + \underset{\substack{\downarrow \\ \text{rotation}}}{Q}x & \det Q = 1 \\ c = f(0) & \text{proper orthogonal rotation} \end{array}}$$

$$\nabla f = Q \qquad \det \nabla f = \det Q > 0$$

$\det Q = 1$ proper orthogonal

$-1 \not> 0$ improper "

What if we represent a rigid motion w.r.t. another coordinate system?



$$\vec{d} = o\bar{o}$$

$$\boxed{\begin{aligned} &① \quad d + \bar{x} = x \\ &② \quad d + \bar{y} = y \end{aligned}}$$

in o coordinate system we have a rigid motion

$$③ \quad y = c + Qx$$

Q: Do we have a rigid motion in $\bar{o}$ coordinate system?

$$②\Rightarrow③ \qquad \bar{y} + d = c + Qx$$

$$① \quad x = d + \bar{x}$$

$$\bar{y} + d = c + Q(d + \bar{x})$$

$$\bar{y} = \underbrace{(c + Qd - d)}_{\bar{c}} + Q\bar{x}$$

$$\left.\begin{aligned} y &= c + Qx \\ o\bar{c} &= d \end{aligned}\right\} \Longrightarrow \underbrace{\begin{aligned} \bar{y} &= \bar{c} + Q\bar{x} \\ \bar{c} &= c + (Q - I)d \end{aligned}}$$

① translation vector depends on coordinate system

② Rotation is independent

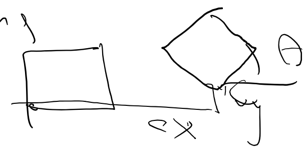
② Rotation is independent of coordinate system

③ A rigid motion in coordinate system is rigid in all coordinate system

In 3D 3 translations & 3 rotations

2D 2 " & 1 "

1D 1 " & 0 rotation

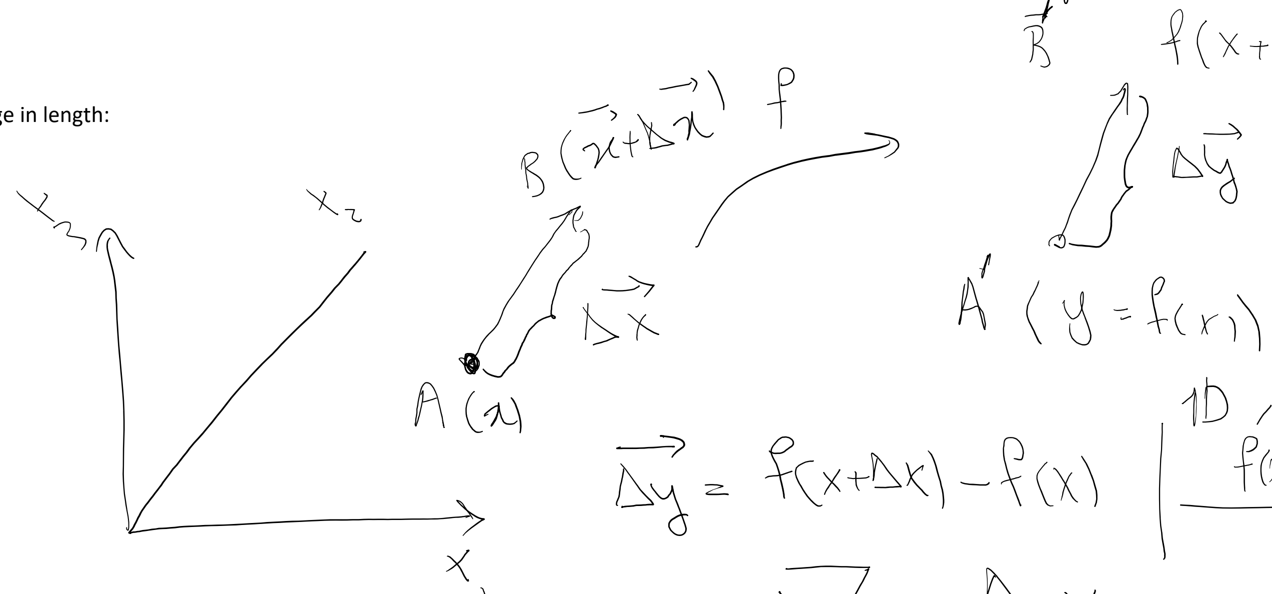


Kinematics: Change in length, angle, area, and volume by deformation
We first study these changes for finite deformation and eventually find their approximate form for infinitesimal deformation

$$F = H + I \qquad F = \nabla y_{/x} \qquad H = \nabla u_{/x}$$

if H is small : infinitesimal deformation theory

1. Change in length:

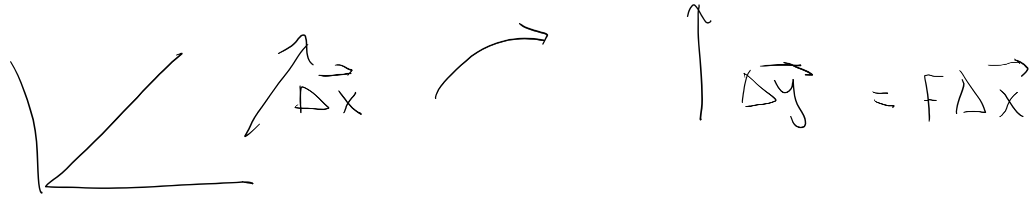


$$\vec{\Delta y} = f(x+\Delta x) - f(x) \qquad \text{1D} \quad f'(x)\Delta x$$

$$= \underbrace{\nabla y_{/x}}_{\text{gradient of } y \,:\, F} \Delta x$$

$$* \quad \boxed{\vec{\Delta y} = F\,\vec{\Delta x}, \text{ as } \vec{\Delta x} \to 0}$$

$$\boxed{\Delta y = F\,\Delta x \quad \text{as } \Delta x \to 0 \qquad F = \nabla_x \underset{\varphi}{y} \Rightarrow F_{ij} = \frac{\partial y_i^{\,f_i}}{\partial x_j}}$$



$$y_1 = 1.1\,x_1$$
$$y_2 = 1.5\,x_2$$
$$y_3 = 1.2\,x_1 + 1.7\,x_3$$

$$\longrightarrow F = \begin{bmatrix} 1.1 & 0 & 0 \\ 0 & 1.5 & 0 \\ 1.2 & 0 & 1.7 \end{bmatrix}$$

$$\begin{bmatrix} \Delta y_1 \\ \Delta y_2 \\ \Delta y_3 \end{bmatrix} = \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \end{bmatrix}$$

$$|\Delta y| = ?$$

$$\vec{u} \qquad |u| = \sqrt{u \cdot u}$$

$$\left.\begin{array}{l} |\Delta y| = \sqrt{\vec{\Delta y \cdot \Delta y}} \\ \Delta y = F\Delta x \end{array}\right\} \Rightarrow |\Delta y| = \sqrt{(F\Delta x)\cdot(F\Delta x)}$$

F^T

$$|\Delta y| = \sqrt{\vec{(\underbrace{F^t F}_{C}\Delta x)\cdot \Delta x}}$$

C Right Cauchy tensor

$$\mathrm{I} \qquad \boxed{\begin{array}{c} |\Delta y| = \sqrt{\Delta x \cdot C \Delta x} \\ C = F^t F \text{ right Cauchy tensor} \end{array}}$$

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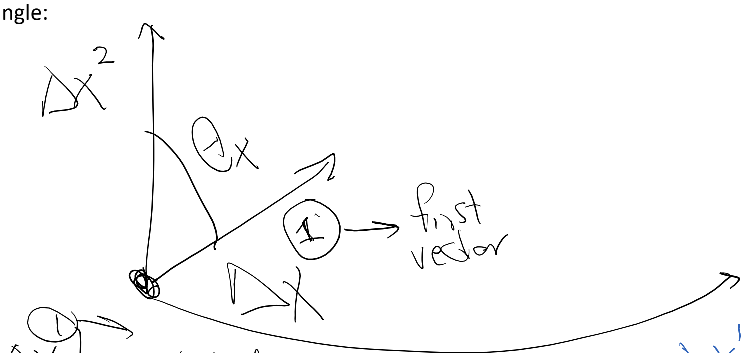
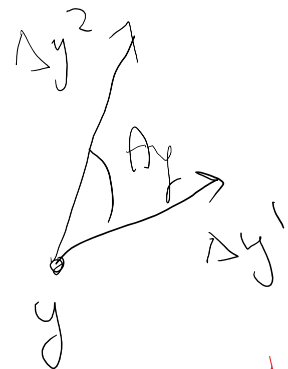
2. Change of angle:





$$\Delta x^1 = \begin{bmatrix} \Delta x_1 \\ \Delta x_2^1 \\ \Delta x_3^1 \end{bmatrix}$$

1 → vector #

2 → coordinate component

$$\cos \theta_x = \frac{\Delta x^1 \cdot \Delta x^2}{|\Delta x^1||\Delta x^2|}$$

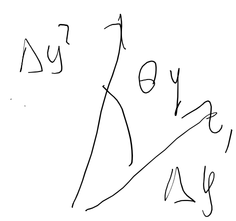
$$\cos \theta_y = \frac{\Delta y^1 \cdot \Delta y^2}{|\Delta y^1||\Delta y^2|}$$

Recall angle between two vectors:

u_1, u_2 $\quad u_1 \cdot u_2 = \cos\theta_{u_1,u_2} |u_1||u_2|$

$$\Longrightarrow \boxed{\cos\theta_{u_1,u_2} = \frac{u_1 \cdot u_2}{|u_1||u_2|}}$$

$$\cos \theta_y = \frac{\Delta y^1 \cdot \Delta y^2}{|\Delta y^1||\Delta y^2|}$$



$\Delta y^1 = F \Delta x^1$
$\Delta y^2 = F \Delta x^2$ $\longrightarrow$ $\Delta y^1 \cdot \Delta y^2 = F\Delta x^1 \cdot F \Delta x^2$

$$= F^t F \Delta x^1 \cdot \Delta x^2$$

also $\quad \Delta x^1 \cdot \underbrace{F^t F}_{C} \Delta x^2$

Right Cauchy Green strain tensor

$\Longrightarrow$

Right Cauchy Green strain tensor

$$|\Delta y^i| = \sqrt{\Delta x^i \cdot C \Delta x^i}$$

no summation on i

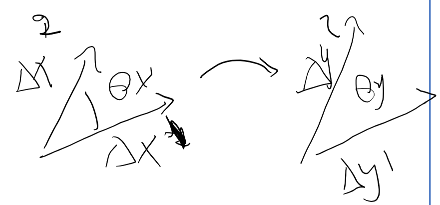
$$\cos(\theta_y) = \frac{\Delta x^1 \cdot C \Delta x^2}{\sqrt{\Delta x^1 \cdot C \Delta x^1}\,\sqrt{\Delta x^2 \cdot C \Delta x^2}}$$

Summary

$$\vec{\Delta y} = F \vec{\Delta x}$$

$$|\vec{\Delta y}| = \sqrt{\Delta x \cdot C \Delta x}$$

$$\cos(\theta_y) = \frac{\Delta x_1 \cdot C \Delta x^2}{\sqrt{\Delta x_1 \cdot C \Delta x_1}\,\sqrt{\Delta x_2 \cdot C \Delta x_2}}$$



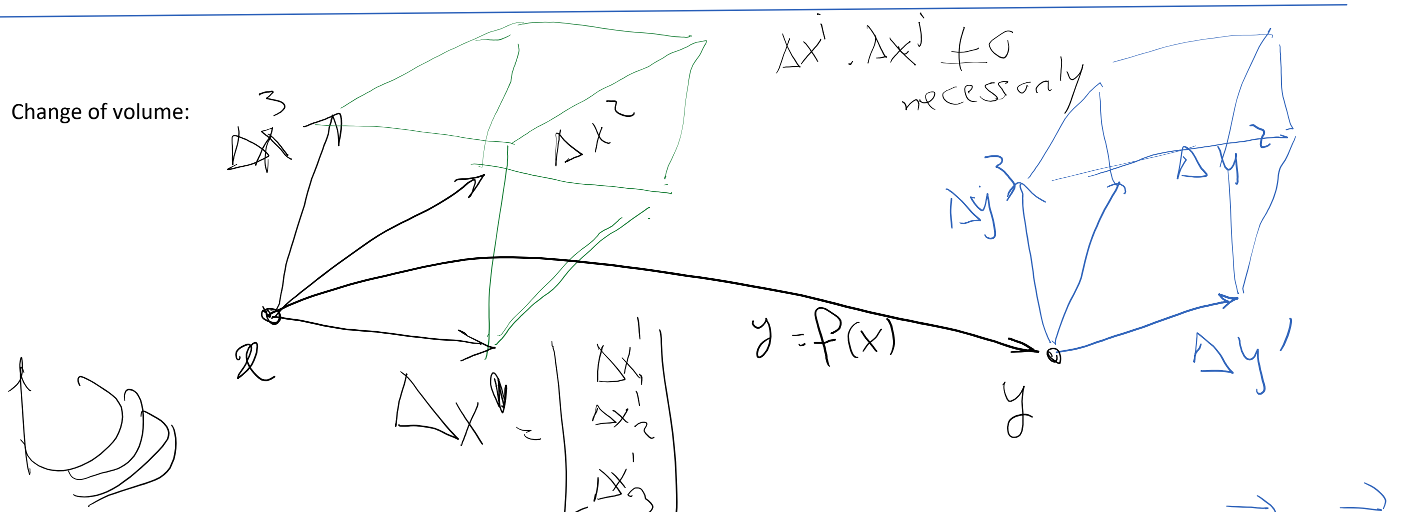
$C = F^t F$ Right Cauchy-Green strain tensor

These equations are appropriate for Lagrangian framework (often used for solids) where x and Delta x are known. In Eulerian we have y and eventually want to find change of length, angle, strains, etc. represented based on current configuration y. In this case, we will be using

$B = F F^t$ Left Cauchy-Green strain tensor

Change of volume:

$\Delta x^i \cdot \Delta x^j \neq 0$ necessarily

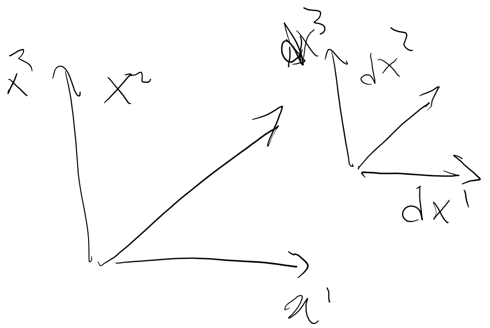


$$\Delta V_x = (\Delta x^1 \times \Delta x^2) \cdot \Delta x^3 \qquad \Delta V_y = (\Delta y^1 \times \Delta y^2) \cdot \Delta y^3$$

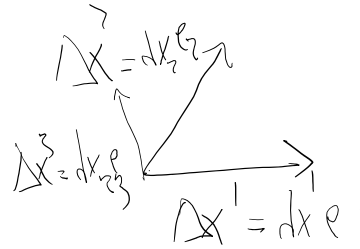
triple product

$$\det \begin{bmatrix} \Delta x^1 \\ \hline \Delta y^2 \\ \hline \Delta x^3 \end{bmatrix} = \det \begin{bmatrix} \Delta x^1_1 & \Delta x^1_2 & \Delta x^1_3 \\ \Delta x^2_1 & \Delta x^2_2 & \Delta x^2_3 \\ \Delta x^3_1 & \Delta x^3_2 & \Delta x^3_3 \end{bmatrix}$$

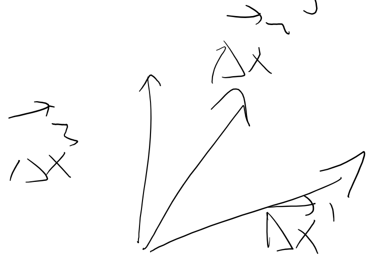
Side note:
We are calculating this change of volume for arbitrary triple of Delta x1, Delta x2, Delta x3.



$$dV_x = dx_1\, dx_2\, dx_3 \equiv$$



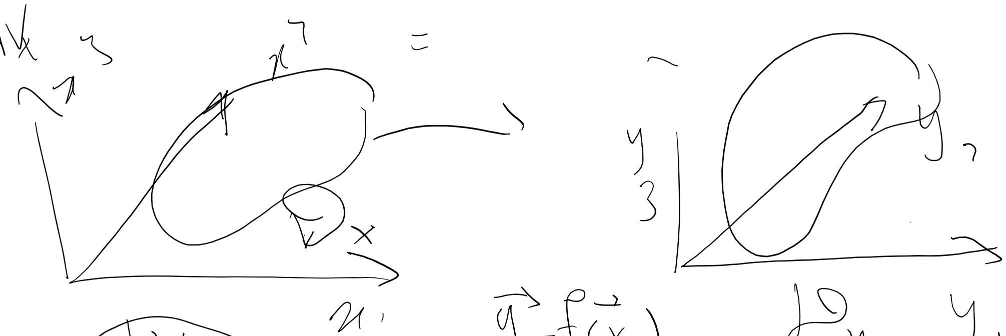
But we are doing it in more general sense



What is ?

$$\Delta V_y = (\cdot)\Delta V_x$$

$$\int_{D_x} f(x_1, x_2, x_3)\, dV_x =$$



$$= \int_{D_y} g(y_1, y_2, y_3)\, dV_y$$

$$\vec{y} = f(\vec{x})$$

$$J\, dV_x \qquad J = \det \nabla y_x$$

From our calculus we expect

$$dV_y = J\, dV_x \qquad J = \det \nabla_{y/x}$$

$$J = \det F$$

Let's prove this relation:

$$\Delta V_x = (\Delta X^1 \times \Delta X^2) \cdot \Delta X^3$$
$$= \epsilon_{ijk} \Delta X^1_i \Delta X^2_j \Delta X^3_k$$

$$\Delta V_y = (\Delta y^1 \times \Delta y^2) \cdot \Delta y^3$$
$$= \epsilon_{ijk} \Delta y^1_i \Delta y^2_j \Delta y^3_k$$

$$\Delta y^i = F \Delta X^i$$

$$\Delta V_y = (?)\ \Delta V_{\cancel{x}}$$

$$\Delta V_y = (F \underbrace{\Delta X^1}_{a}) \times (F \underbrace{\Delta X^2}_{b}) \cdot (F \underbrace{\Delta X^3}_{c})$$
$$= \det F\ \underbrace{(\underbrace{\Delta X^1}_{a} \times \underbrace{\Delta X^2}_{b}) \cdot \underbrace{\Delta X^3}_{c}}_{\Delta V_x} = (\det F)\ \Delta V_x$$

Why $(Fa \times Fb) \cdot Fc = \det F\, (a \times b) \cdot c$?

$$(Fa \times Fb) \cdot Fc = \epsilon_{ijk} (Fa)_i (Fb)_j (Fc)_k$$
$$= \epsilon_{ijk} (F_{im} a_m)(F_{jn} b_n)(F_{kp} c_p)$$
$$= \underbrace{(\epsilon_{ijk} F_{im} F_{jn} F_{kp})}\ a_m b_n c_p$$
$$(\det F\ \epsilon_{mnp})\ a_m b_n c_p$$
$$= \det F\ (\underbrace{\epsilon_{mnp} a_m b_n c_p})$$
$$= \det F\ ((a \times b) \cdot c)$$

We proved $\{(Fa)\times(Fb)\}\cdot Fc = \det F\{(a\times b)\cdot c\}$

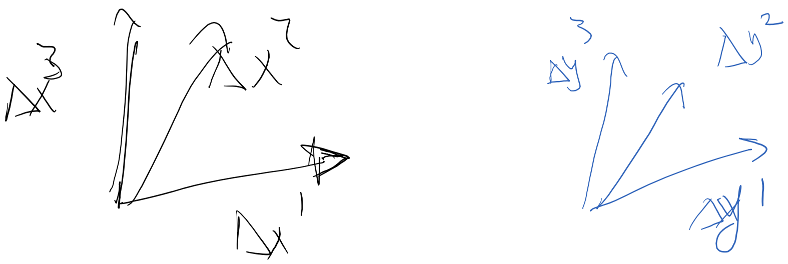
$a=\Delta x^1$, $b=\Delta x^2$, $c=\Delta x^3$

$Fa=\Delta y^1$, $Fb=\Delta y^2$, $c=\Delta y^3$

$\Longrightarrow$

$$(\Delta y^1\times\Delta y^2)\cdot\Delta y^3=\det F(\Delta x^1\times\Delta x^2)\cdot\Delta x^3$$

$$\boxed{\Delta V_y=\det F\,\Delta V_x}$$



$\Delta V_x>0$ $\qquad$ $\Delta V_y>0$ $\longrightarrow$ $\det F>0$

Definition 72 *Let* $\overset{0}{\mathcal{B}}$ *be an open, bounded, regular region of a Euclidean point space* $\mathcal{E}$. *A* deformation $\mathbf{f}$ *is a mapping (function) of points in* $\overset{0}{\mathcal{B}}$ *onto another open region of* $\mathcal{E}$ *with the properties*

1. $\mathbf{f}$ *is one-to-one; i.e.,* $\mathbf{f}(\mathbf{x})=\mathbf{f}(\mathbf{y})\Rightarrow\mathbf{x}=\mathbf{y}\ \forall\ \mathbf{x},\mathbf{y}\in\overset{0}{\mathcal{B}}$,
2. $\mathbf{f}\in C^2(\overset{0}{\mathcal{B}})$, $\mathbf{f}^{-1}\in C^2(\mathbf{f}(\overset{0}{\mathcal{B}}))$,
3. $\det\nabla\mathbf{f}(\mathbf{x})>0\ \forall\ \mathbf{x}\in\overset{0}{\mathcal{B}}$.

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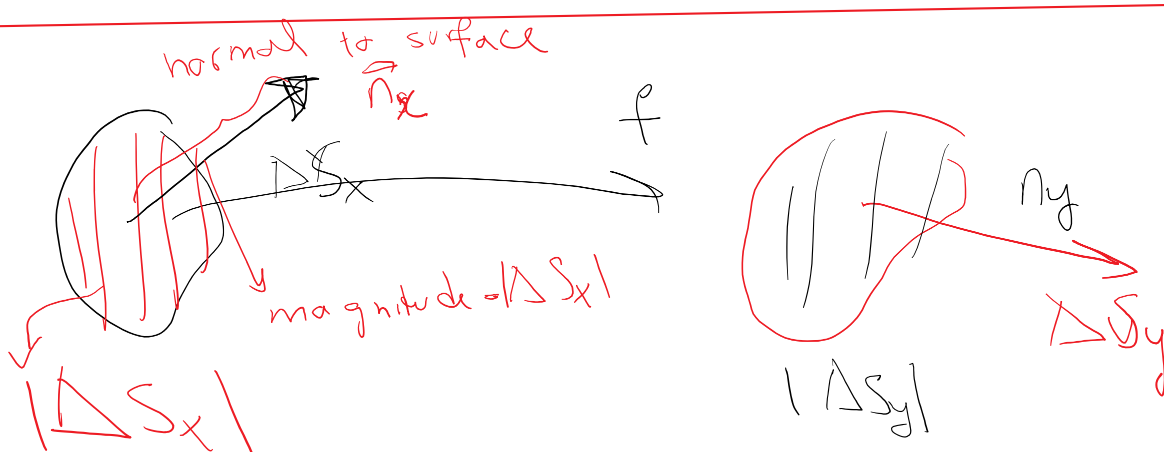
4. Change of areas:



$\Delta S_x=|\Delta S_x|n_x$ $\qquad$ $\Delta S_y=|\Delta S_y|\vec{n}_y$

both magnitude & orientation both can change

$|\Delta S_y| \neq |\Delta S_x|$

$\vec{n}_y \neq \vec{n}_x$

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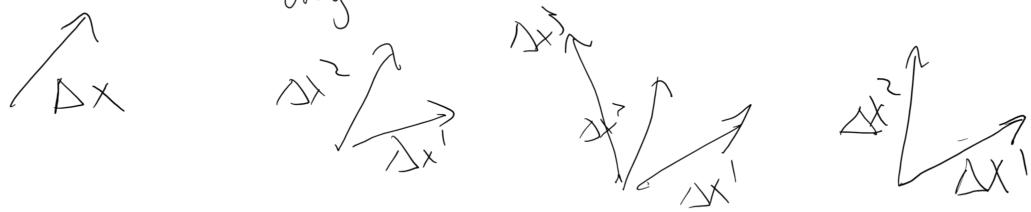
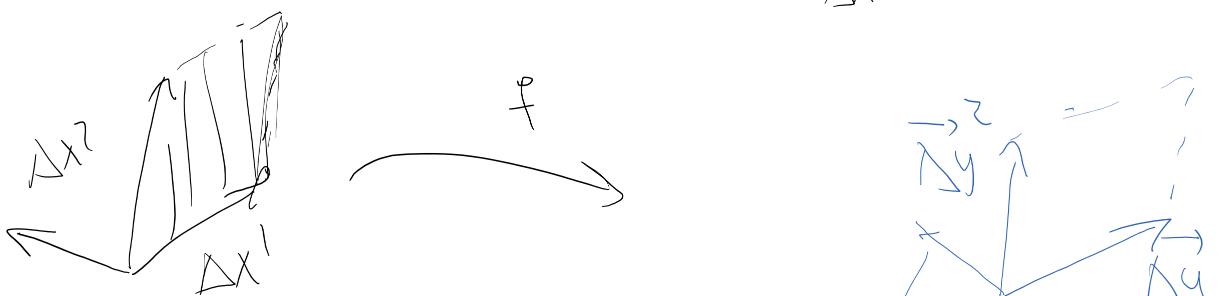
length angle volume Surface





$$\vec{\Delta S}_x = \vec{\Delta x}^1 \times \vec{\Delta x}^2 \qquad \vec{\Delta S}_y = \vec{\Delta y}^1 \times \vec{\Delta y}^2$$

$$\underbrace{\vec{\Delta S}_y}_{\text{vector}} = \underbrace{\left(\overset{?}{} \right)}_{\text{2nd order tensor}} \underbrace{\vec{\Delta S}_x}_{\text{vector}}$$

$$\vec{\Delta S}_y = (\Delta S_y)_1 e_1 + (\Delta S_y)_2 e_2 + (\Delta S_y)_3 e_3$$

$$(\Delta S_y)_1 = (\vec{\Delta S}_y) \cdot e_1$$

or in general

$$(\vec{\Delta S}_y)_i = (\Delta S_y) \cdot e_i$$

$$\Delta y \left(\quad {}^{(1)} \times \Delta y^2 \right) \cdot e_i$$

$$= (F\Delta x^1 \times F\Delta x^2) \cdot \underbrace{e_i}_{F=?}$$

$$(Fa \times Fb) \cdot Fc = \det F (a \times b) \cdot c$$

$$= (F\Delta x^1 \times F\Delta x^2) \cdot (\underbrace{F F^{-1}}_{I} e_i)$$

$$= (F\underbrace{\Delta x^1}_{a} \times F\underbrace{\Delta x^2}_{a}) \cdot F(\underbrace{F^{-1} e_i}_{c})$$

$$= \det F \; \underbrace{(\Delta x^1 \times \Delta x^2)} \cdot F^{-1} e_i$$

$$= \det F \; \Delta S_x \cdot F^{-1} e_i$$

$$= \det F \left((F^{-1})^t \Delta S_x \right) \cdot e_i$$

$$V \cdot e_i = V_i$$

$$\boxed{(\Delta S_y)_i = (\det F) \left(F^{-t} \Delta S_x \right)_i}$$

$$\Rightarrow \quad \overrightarrow{\Delta S_y} = \underbrace{(\det F \, F^{-t})}_{\text{2nd order tensor that maps areas}} \overrightarrow{\Delta S_x}$$

Another proof of this will be given next time

$$\boxed{\begin{aligned} \Delta y &= F \Delta X \\ |\Delta y| &= \sqrt{\Delta X \cdot C \Delta X} \\ \cos\theta_y &= \frac{\Delta X_1 \cdot C \Delta X_2}{\sqrt{\Delta X_1 \cdot C \Delta X_1}\,\sqrt{\Delta X_2 \cdot C \Delta X_2}} \end{aligned}}$$

1 component of F

$$F = \begin{bmatrix} 1.2 & & \\ & 1.2 & \\ & & 1.2 \end{bmatrix}$$

$$\Delta y = 1.2 \, \Delta x$$

$$\cos\theta_y = \frac{\ldots}{\sqrt{\Delta x_1 \cdot C\,\Delta x_1}\;\sqrt{\Delta x_2 \cdot C\,\Delta x_2}}$$

$$\boxed{\Delta V_y = \det \underline{F}\;\Delta V_x}$$

$$\boxed{\Delta \vec{S}_y = \left(\det F\, F^{-t}\right) \Delta S_x}$$

$\Delta y = 1.2\,\Delta x$

$\longrightarrow \Delta V_y = (1.2)^3\,\Delta V_x$

$\Delta \vec{S}_y = (1.2)^2\,\Delta V_x$

for this simple hydrostatic deformation