2019/10/07

Monday, October 7, 2019 11:29 AM

$$\vec{\Delta S_y} = \det F \, F^{-T} \vec{\Delta S_x}$$ Change of surface

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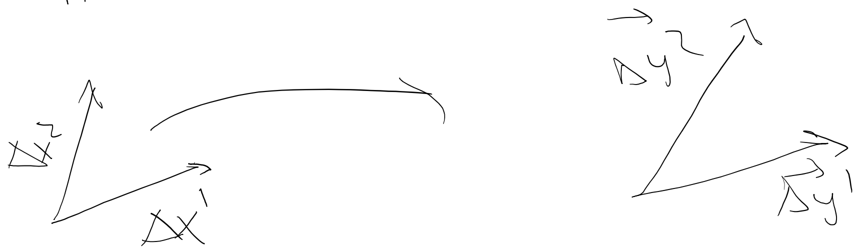
Another approach to prove this



$$\vec{\Delta S_y} = \Delta y^1 \times \Delta y^2 = \epsilon_{ijk} \, \Delta y^1_i \, \Delta y^2_j \, e_k$$

$$\left.\begin{array}{l} \Delta y^1 = F \Delta x^1 \implies \Delta y^1_i = F_{im} \Delta x^1_m \\ \Delta y^2 = F \Delta x^2 \implies \Delta y^2_j = F_{jn} \Delta x^2_n \end{array}\right\} \implies$$

$$\Delta S_y = \epsilon_{ijk} (F_{im} \Delta x^1_m) (F_{jn} \Delta x^2_n) \, e_k =$$

$$\left.\begin{array}{l} = \underbrace{\epsilon_{ijk} F_{im} F_{jn}} \Delta x^1_m \Delta x^2_n \, e_k \\ \epsilon_{ijk} F_{im} F_{jn} = |\det F| \epsilon_{mnp} F^{-1}_{pk} \end{array}\right\} \implies$$

$$\Delta S_y = \epsilon_{mnp} F^{-1}_{pk} \Delta x^1_m \Delta x^2_n \, e_k \det F = \underbrace{(\epsilon_{mnp} \Delta x^1_m \Delta x^2_n)}_{\text{component } p \text{ of } \Delta x^1 \times \Delta x^2} \underbrace{F_{pk}}_{} e_k$$

det T

$$= \det F \, F^{T}_{kp} \underbrace{(\Delta x^1 \times \Delta x^2)_p}_{(\vec{\Delta S_x})_p} \, e_k$$

$$\Delta S_y = \det F (F^{-T} \Delta S_x)_k \, e_k \implies \boxed{\vec{\Delta S_y} = \det F \, F^{-t} \vec{\Delta S_x}}$$

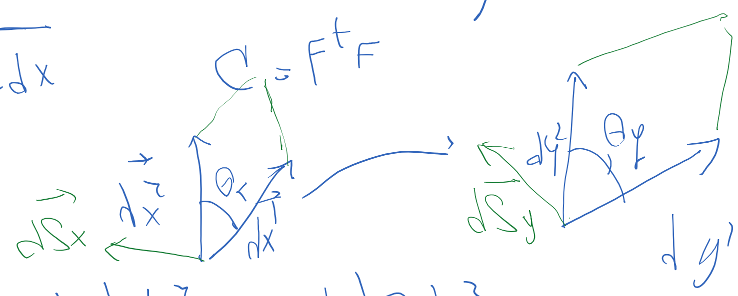
Summary of all kinematic relations:

$dy = F dx$ $\quad \vec{dx} \quad \longrightarrow \quad \vec{dy} = F \vec{dx}$

$F^t F$

$$|dy| = \sqrt{dx\, C\, dx} \qquad C = F^t F$$

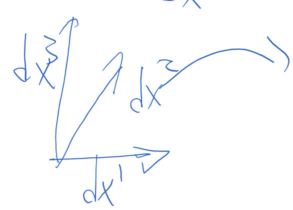
angle



$$\cos(\theta_y) = \frac{dy^1 \cdot dy^2}{|dy^1|\,|dy^2|} = \frac{dx^1 C\, dx^2}{\sqrt{dx^1 C\, dx^1}\,\sqrt{dx^2 C\, dx^2}}$$

area $\quad d\vec{S}_y = \det F\, F^{-t}\, d\vec{S}_x$

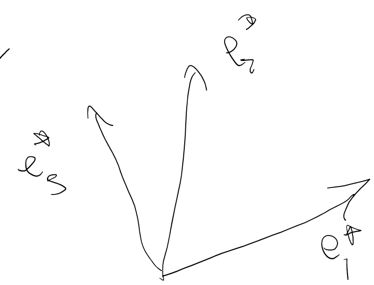
volume $\quad dV_y = \det F\, dV_x$



Understanding the effect of C

$C = F^t F$ right deformation tensor

positive definite



e_1^*, e_2^*, e_3^* are principal axes of C

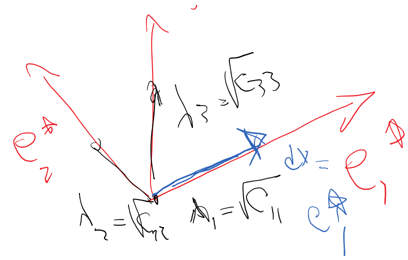
$$[C]^* = \begin{bmatrix} C_{11}^* & 0 & 0 \\ 0 & C_{22}^* & 0 \\ 0 & 0 & C_{33}^* \end{bmatrix} \qquad U = \sqrt{C}$$

$$[U]^* = \begin{bmatrix} U_{11}^* & 0 & 0 \\ 0 & U_{22}^* & 0 \\ 0 & 0 & U_{33}^* \end{bmatrix} \qquad U_{\underline{ii}} = \sqrt{C_{\underline{ii}}} \quad \text{no summation}$$

$$|\vec{dy}| = \sqrt{dx \cdot C dx} = \sqrt{dx \cdot U^2 dx} = \sqrt{dx \cdot U(U dx)} = \sqrt{\underbrace{U^t}_{\text{symmetric}} dx \cdot U dx}$$

$$= \sqrt{(U dx)\cdot(U dx)} = |U dx|$$

$$|dy| = \sqrt{dx \cdot C\, dx} = |U\, dx|$$



$$U^* = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}$$

$$dx^* = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow dy^* = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \lambda_1$$

in e_1^* $\longrightarrow$ λ_1 ratio of $|dy|/|dx|$

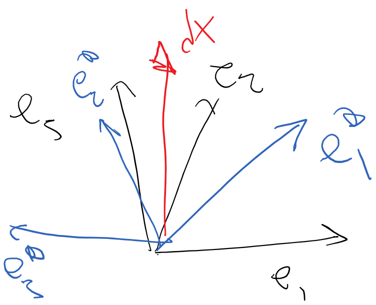
in e_2^* $\quad$ λ_2 "

in e_3^* ($\longrightarrow$ λ_3 "

along the principal directions of $C(U)$ the orientation of dx & dy are the same

U represents stretch $\longrightarrow$ U : right stretch tensor

$C = \sqrt{F^t F}$ right deformation tensor



$$\underbrace{U \vec{dx}}$$

$\downarrow$

general change the angle of $\vec{dx}$

$$\begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix} \begin{bmatrix} dx_1^* \\ dx_2^* \\ dx_3^* \end{bmatrix} = \begin{bmatrix} \lambda_1\, dx_1 \\ \lambda_2\, dx_2 \\ \lambda_3\, dx_3 \end{bmatrix} \parallel \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

in general except when $\lambda_1 = \lambda_2 = \lambda_3$

$$F = RU \qquad \text{polar decomposition}$$

$$dy = F\, dx = RU\, dx = \underbrace{R}\,\overbrace{(\underbrace{U dx})}$$

rotation part

stretching part

$|dy| = |U dx|$

rotates dy

Definition 80 *Let the deformation gradient* $\mathbf{F} = \nabla \mathbf{f}$ *of the deformation* $\mathbf{f}$ *of* $\overset{0}{\mathcal{B}}$ *have the polar decompostion*

$$\mathbf{F}(\mathbf{x}) = \underbrace{\mathbf{R}(\mathbf{x})\mathbf{U}(\mathbf{x})} = \underbrace{\mathbf{V}(\mathbf{x})\mathbf{R}(\mathbf{x})}$$

$\forall\, \mathbf{x} \in \overset{0}{\mathcal{B}}$, *where* $\mathbf{U}(\mathbf{x}), \mathbf{V}(\mathbf{x}) \in \mathrm{Psym}$ *and* $\mathbf{R}(\mathbf{x}) \in \mathrm{Orth}\,\mathcal{V}^{+}$. *The following terminology is standard.*

- $\mathbf{R}(\mathbf{x})$ — *the* rotation tensor *at* $\mathbf{x}$;
- $\mathbf{U}(\mathbf{x})$ — *the* right stretch tensor *at* $\mathbf{x}$;
- $\mathbf{V}(\mathbf{x})$ — *the* left stretch tensor *at* $\mathbf{x}$;
- $\mathbf{C}(\mathbf{x}) = \mathbf{F}^t(\mathbf{x})\mathbf{F}(\mathbf{x})$ — *the* right Cauchy-Green deformation tensor *at* $\mathbf{x}$;
- $\mathbf{B}(\mathbf{x}) = \mathbf{F}(\mathbf{x})\mathbf{F}^t(\mathbf{x})$ — *the* left Cauchy-Green deformation tensor *at* $\mathbf{x}$.

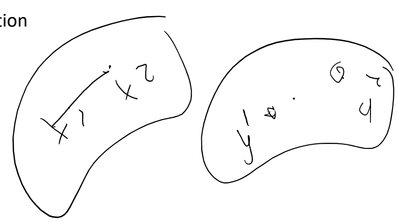
$F = RU = VR$

R is a rotation

$\det F > 0$

V has the role of U but in an Eulerian view point where $\vec{dy}$ is given & want to see what dx it corresponds to & what is the corresponding stretch

Reminder: Rigid deformation



$|y^1 - y^2| = |x^1 - x^2|$

$\Downarrow$

$y = \underset{\sim}{Q} x + c$

rotation, translation

how about a general deformation?

- do we still have a c & Q (pointwise)
- what else do we have for a local deformation

$F = RU$

rotation but R(x) changes point to point in general rotation but $R(X)$ changes point to point in general

for rigid motion $U = I$

why? $y = Qx + c \qquad F = \nabla_X y = Q I = R U$

for rigid motion $R = Q$, $U = I$ no stretching

$$\boxed{C = F^t F = Q^t Q = I}$$

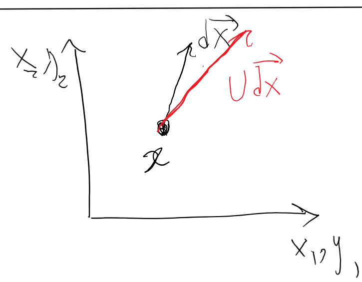
for rigid motion

For general deformation: for each point we have 3 different deformation components

- translation
- stretching
- rotation

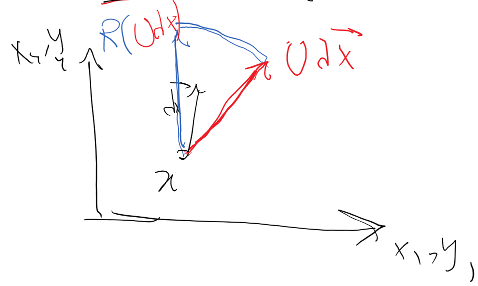
A Right path

1. Stretch



$(x, dX) \rightarrow (x, U dX)$

2. a rigid rotation



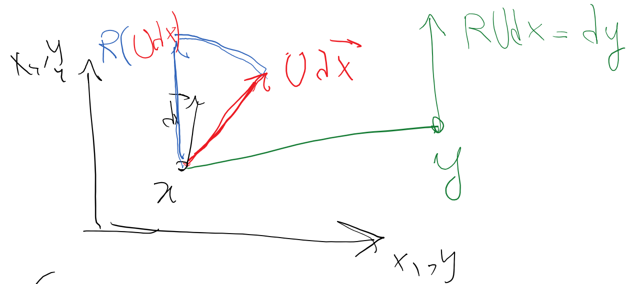
$(x, U dX) \rightarrow (x, \underbrace{RU}_{F} dX) = F dX$

$(x, U dx) \to (x, \underbrace{RU}_{F} dx)$

$\underbrace{F dx}_{dy}$

$(x, dx) \to (x, U dx) \to (x, dy)$

3. a rigid translation



1 sketch U → 2 rotation → 3 translation

$(x, dx) \longrightarrow (y, dy)$

$R(X)$

pure rigid motion $R(x) = \text{constant} = Q$

right path $\longleftrightarrow$ $C = F^t F$ right deformation tensor

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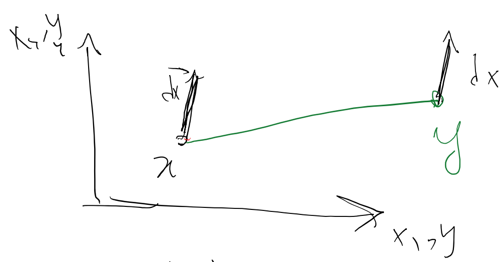
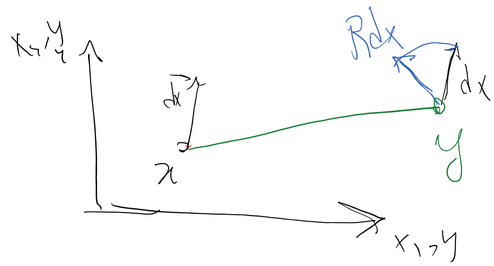
Left path $F = RU = VR$

1. Rigid translation



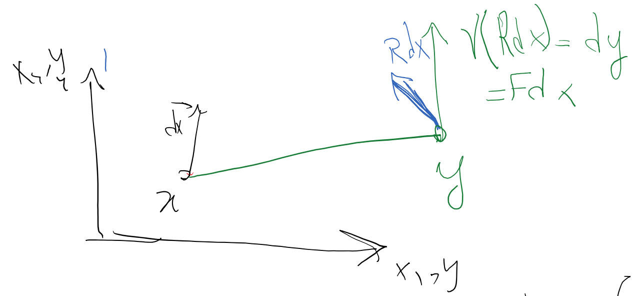
$(x, dx) \longrightarrow (y, dx)$

2. Rotation



$(y, dx) \longrightarrow (y, \; Rdx)$

3. Stretch:



$(y, Rdx) \longrightarrow (y, \underbrace{VR}_{F}dx)$

$(\cancel{y}, dx) \underset{\text{translation}}{\longrightarrow} (y, dx) \longrightarrow (y, Rdx) \underset{\text{rotation}}{} \longrightarrow (y, dy = VRdx)$ stretch

Left path

$$F = R\underbrace{U = V}R \qquad RU \neq UR$$

$$U \neq V \qquad \text{why?}$$

$$U = \sqrt{\underbrace{F^tF}_{C}} \qquad V = \sqrt{\underbrace{FF^t}_{B}} \qquad \underbrace{AB \neq BA}$$

that's why $RU \neq UR$

we have different stretches

note

Small deformation gradient

$$H = \nabla u_{/x} = \underbrace{\left(\frac{H + H^t}{2}\right)}_{\varepsilon} + \underbrace{\left(\frac{H - H^t}{2}\right)}_{W}$$

strain — similar to U

W infinitesimal rotation — similar R

$A + B = B + A$ ✓

In small deformation stretch change (E) & rotation change (w) are interchangeable.

Theorem 128:

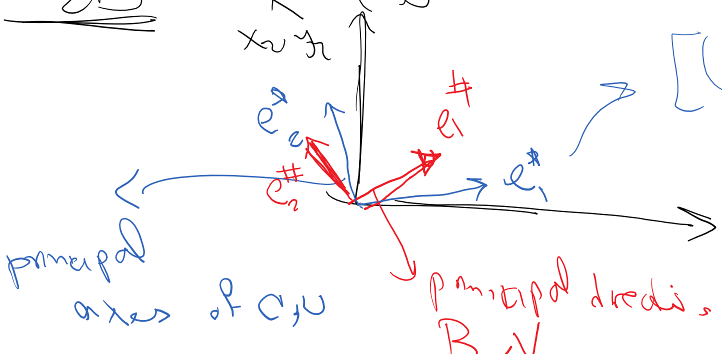
1. $C = U^2 \qquad B = V^2$
2. $\boxed{V = RUR^T}$ — V is the rotation of tensor U by R — $U = R^T V R$
3. $\boxed{B = RCR^T}$ — B rotation of C by R — $C = R^T B R$

Inverse rotation.

Hint, HW2

$T'_{ij} = Q_{im} Q_{jn} T_{mn} \qquad [T'] = \underline{Q T Q^T}$

2D $\quad R$ (30° rotation)



$$[U]^* = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}$$

same representation in different coordinate system

$$[V]^{\#} = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}$$

$y = X + U \rightarrow$ displacement

Some other points about rigid deformation:

$C = F^T F \qquad \underline{F = \nabla y_X} = \nabla_{X+U_X} = H + I$, $\quad H = \nabla U_X$

$F = H + I$

$C = (H+I)^T(H+I) = (H^T+I)(H+I) = \underbrace{H^T H + H + H^T}_{} + I$

represents change of dy from dx

Def. 81

$$G = \frac{1}{2}(C - I) = \frac{1}{2}(H + H^t + H^t H)$$

$$= E + \frac{1}{2} H^t H$$

Green St Venant strain.

$E = \frac{1}{2}(H + H^t)$ small def. gradient strain tensor

infenitisimal strain tensor

$$C_{ij} = (F^t F)_{ij} = F^t_{im} F_{mj} = F_{mi} F_{mj}$$

$$F_{mi} \cdot \frac{\partial x_m}{\partial X_i} = \frac{\partial (X_m + U_m)}{\partial X_i} = \underbrace{\delta_{mi}}_{\frac{\partial X_m}{\partial X_i}} + \underbrace{\frac{\partial U_m}{\partial X_i}}_{H_{mi}}$$

$$F_{mi} = \delta_{mi} + H_{mi}$$

$$F_{mj} = \delta_{mj} + H_{mj} \implies$$

$$C_{ij} = (\delta_{mi} + H_{mi})(\delta_{mj} + H_{mj})$$

$$\underbrace{\delta_{mi}\delta_{mj}}_{\delta_{ij}} + \delta_{mi} H_{mj} + \delta_{mj} H_{mi} + H_{mi} H_{mj}$$

$$C_{ij} = \delta_{ij} + H_{ij} + H_{ji} + H_{mi} H_{mj}$$

$C = (H+I)^t (H+I)$

$$G_{ij} = \underbrace{\left(\frac{H_{ij} + H_{ji}}{2}\right)}_{E_{ij}} + \frac{1}{2} H_{mi} H_{mj}$$

$G = \frac{1}{2}(C - I)$

$= \frac{1}{2}(H + H^t + H^t H)$

$G = \frac{1}{2}(C - I)$

$$E_{ij} = \frac{H_{ij} + H_{ji}}{2}$$

$$E = \frac{H + H^t}{2}$$

Rigid deformation

$$y = Qx + c \Rightarrow F = Q \Rightarrow C = F^t F = Q^t Q = I$$

$$C = I$$

rigid motion
difficult

$$C = I \iff G = \frac{1}{2}(C - I) = 0$$

strain $= 0$
should mean rigid motion

$$G = E + \frac{1}{2} H^t H$$

$$\Downarrow$$

$$0 \qquad E = -\frac{1}{2} H^t H \neq 0$$