2019/10/09

Wednesday, October 9, 2019 11:31 AM

Large deformation (no approximation)

Infinitesimal displacement (uses u)

$dy = F\,dx$

$F = \underbrace{R}_{\text{rotation}}\underbrace{U}_{\text{stretch}} = VR$

$u = y - x$
displacement

$du = H\,dx$

$H = \nabla_X u = F - I$

$du = (E + W)\,dx$

$E = \frac{H + H^T}{2}$ infinitesimal strain

W : infinitesimal rotation

$R \longleftrightarrow W$

exact rotation

infinitesimal rotation

Rigid rotation

large def. gradient

infinitesimal

$U = I$

$E = 0$

$F = R$ rotation

$H = W$ approximate

Constant (not a function of X)

Rigid motion

① x^1, x^2 $\quad |x^1 - x^2| = |y^1 - y^2|$

$y^i = f(x^i)$

$\Updownarrow$

② $f(x) = \underbrace{Q}_{\text{rotation}} x + \underbrace{c}_{\text{translation}}$

$\Updownarrow$

③ $C = I \qquad C = F^t F = Q^t Q = I$

$\Updownarrow$

④ $G = 0 \qquad G = \frac{1}{2}(C - I) \quad \not\Longrightarrow \quad E = 0$

$H = W$

$E = \underline{O(\varepsilon^2)}$

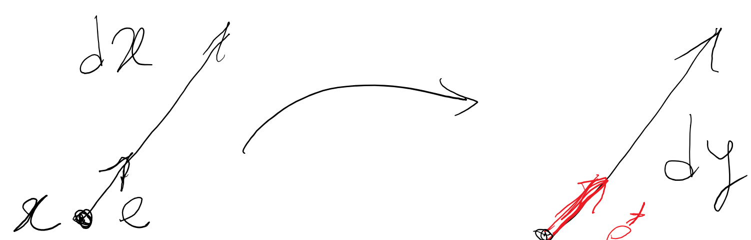
ε is measure of H

$G = O(\varepsilon^2)$

ε is measure of $\underline{H}$

Strains:

A. Normal strain



Goal: what is the relative change of length along direction e:

$$|dy| = \sqrt{dx \cdot C dx} \qquad dx = |dx| e$$

$$\varepsilon(x, \vec{e}) = \frac{|dy| - |dx|}{|dx|} = \frac{\sqrt{(|dx| e) \cdot C(|dx| e)} - |dx|}{|dx|} \qquad \sqrt{e \cdot Ce} - 1$$

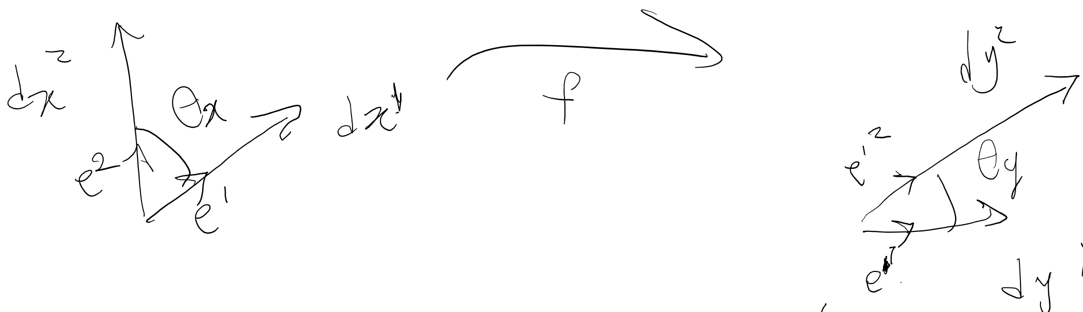
(1) $\boxed{\mathcal{E}(x,\vec{e}) = e\cdot Ce - 1}$ Lagrangian (C, U)

x, e in $\vec{x}$ coordinate system

HW $\mathcal{E}(y, e') = e'\cdot Be$ $B = FF^t$ Left deformation tensor

Eulerian (B, V)

B. Shear strain (Changing angle between two directions)



$\cos\theta_x = e_1\cdot e_2$ $\cos\theta_y = e'^1\cdot e'^2$

$$\cos\theta_y = \frac{dy^1\cdot dy^2}{|dy^1||dy^2|} \qquad dy^1 = F dx^1,\quad dy^2 = F dx^2 \qquad = \frac{dy^1\cdot dy^2}{|dy^1||dy^2|}$$

$$= \frac{F dx^1\cdot F dx^2}{\sqrt{F dx^1\cdot F dx^1}\sqrt{F dx^2\cdot F dx^2}} = \frac{dx^1\cdot \overbrace{F^t F}^{C} dx^2}{\sqrt{dx^1\cdot \underbrace{F^t F}_{C} dx^1}\sqrt{dx^2\cdot \underbrace{F^t F}_{C} dx^2}}$$

$$\boxed{\cos(\theta_y) = \frac{dx^1\cdot C dx^2}{\sqrt{dx^1\cdot C dx^1}\sqrt{dx^2\cdot C dx^2}} = \frac{e^1\cdot Ce^2}{\sqrt{e^1\cdot Ce^1}\sqrt{e^2\cdot Ce^2}}}$$

$\downarrow$

$\theta_y(x, e^1, e^2)$

(2)

$$= \frac{(|dx^1|e^1)\cdot C(|dx^2|e^2)}{\sqrt{(|dx^1|e^1)\cdot C(|dx^1|e^1)}\sqrt{|dx^2|e^2\cdot C|dx^2|e^2}}$$

$v \quad v e \cdot e$ $\quad e'$ $\quad e'$

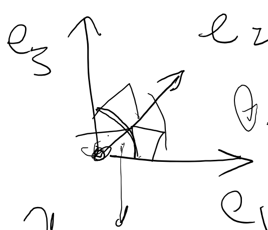
Concept of change $\longrightarrow$ strain will be discussed next

π in values

$$\frac{e'^1 \cdot B e'^2}{\sqrt{e'^1 \cdot B e'^1}\,\sqrt{e'^2 \cdot B e'^2}}$$

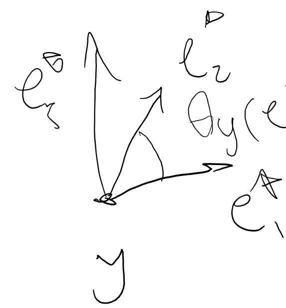
Idea: Represent normal and shear strains in one coordinate system to encode material deformation at that location?
Q: Is our representation going to be a tensor, what is coordinate transformation rule, etc.?

$e_i \cdot e_j = \delta_{ij}$ orthonormal coordinate system



$\theta_x(e_1, e_2) = \frac{\pi}{2}$

$\theta_x(e_1, e_3) = \frac{\pi}{2}$



3 normal strains

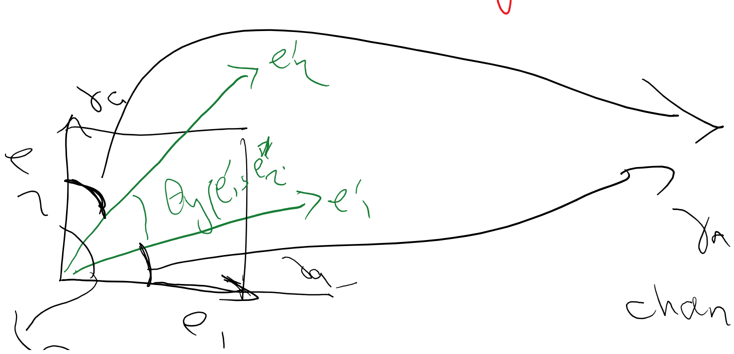
Change in length $\varepsilon(x, e_i) = \sqrt{e_i C \cdot e_i} - 1$

no summation on i

3 shear strains

$\theta_x(e_i, e_j) = \frac{\pi}{2} \quad i \neq j$

$$\underbrace{\gamma_{12}}_{\text{engineering shear strain}} = \theta_x(e_1, e_2) - \theta_y(e'_1, e'_2)$$



$\gamma_a + \gamma_b = \gamma_{12}$

change of angle

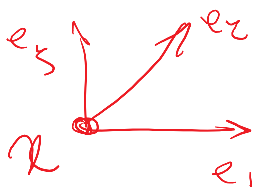
$\theta_x(e_1,e_2) = \frac{\pi}{2}$

$$\boxed{\begin{aligned}\gamma_{12} &= \theta_x(e_1,e_2) - \theta_y(e_1',e_2') \\ &= \frac{\pi}{2} - \theta_y\end{aligned}}$$

$$\sin\gamma_{12} = \sin\left(\frac{\pi}{2} - \theta_y\right)$$

$$= \cos\theta_y = \frac{e_i \cdot C e_j}{\sqrt{e_i \cdot C e_{\underline{i}}}\,\sqrt{e_j \cdot C e_{\underline{j}}}}$$

Summary



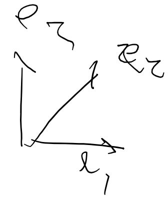
$$\varepsilon_{i\underline{i}}(x) = \varepsilon(x,e_i) = \sqrt{e_i \cdot C e_{\underline{i}}} - 1$$ normal strain

$$\sin\gamma_{ij} = \frac{e_i \cdot C e_j}{\sqrt{e_i \cdot C e_{\underline{i}}}\,\sqrt{e_j \cdot C e_{\underline{j}}}} \longrightarrow \gamma_{ij} = \sin^{-1}\ldots$$

Can I define a strain tensor from these?

$$\varepsilon = \begin{bmatrix} \varepsilon_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{12} & \varepsilon_{22} & \gamma_{23} \\ \gamma_{13} & \gamma_{23} & \varepsilon_{33} \end{bmatrix}$$

2 indexed array



ε'

array

↓ ~~to~~ to be tensor

$\varepsilon'_{ij} = Q_{im} Q_{jn} \varepsilon_{mn}$ ✗

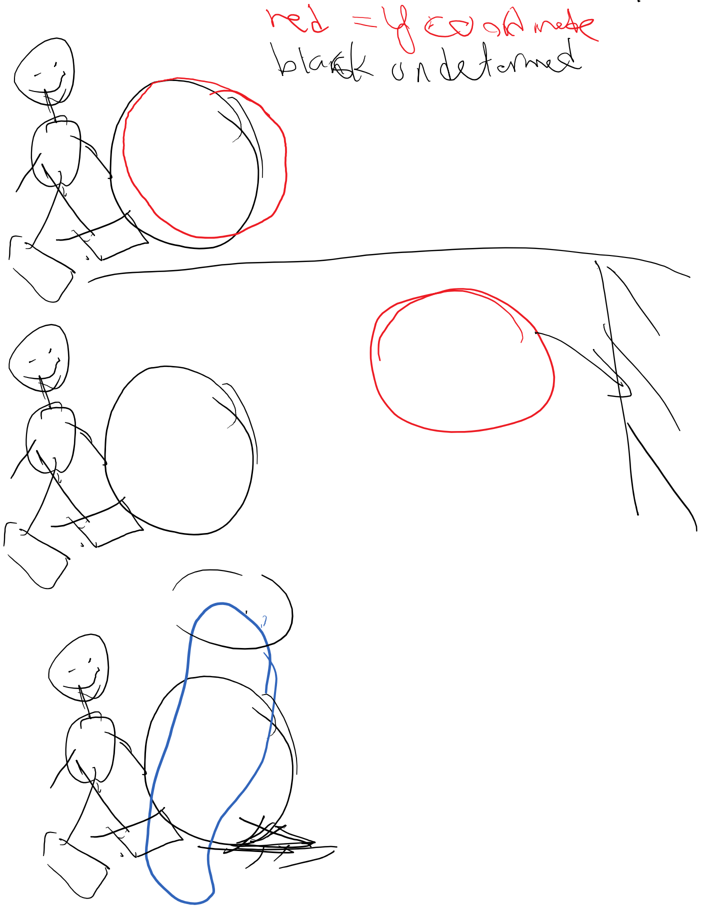
Not a tensor!

Infinitesimal deformation theory

$y = f(x) \rightarrow$

$dy = F\,dx$

$$\boxed{\begin{array}{ll} u = f(x) - x & \\ \text{displacement} & \\ H = \nabla u_x & \text{displacement gradient} \end{array}}$$



small u? small $H = \nabla u_x$ ✗

small | small

large | small

challenging aspect finding contact point

$H = \underbrace{E}_{\text{strain}} + W$

small ?

large

C, U

or

R, V

$\mathbb{B}, V$

$G \quad F = CU = VB$

$e_{ii} = \sqrt{e_i \cdot C e_i}$

difficult

The main challenge is when H is large

$$\boxed{\begin{array}{l} H \text{ large} \longrightarrow \underline{\text{finite}} \text{ strain theory} \\ \qquad\qquad\qquad\quad \text{up to this point} \\ H \text{ small} \longrightarrow \underline{\text{infinitesimal}} \text{ strain theory} \end{array}}$$

How do we classify H being large or small?

$$H_{ij} = \frac{\partial u_i}{\partial x_j} \qquad H = \begin{bmatrix} H_{11} & H_{12} & \\ & & \\ & & H_{33} \end{bmatrix} = \begin{bmatrix} u_{1,1} & \cdot & u_{1,3} \\ & & \\ & & u_{3,3} \end{bmatrix}$$

Φ scalar $\quad \phi < \varepsilon \ll 1$

V vector $\quad |v| = \sqrt{v_i v_i} < \varepsilon \ll 1 \quad l_2 \longrightarrow$ scalar

good $\quad \underline{\operatorname{Max}(|v_i|) < \varepsilon} \quad l_\infty \quad$ changes by coordinate system

$\frac{1}{3} \operatorname{Max}|v_i'| \quad \operatorname{Max}|v_i| \leq 3 \operatorname{Max}|v_i'|$

$$\|H\| : \begin{cases} \sqrt{H:H} & \textcircled{1} \quad \text{Scalar norm of } H \\ \sqrt{H_{ij} H_{ij}} & \\ \max(|\lambda_i|) & \text{eigen values of } H \quad = \max_{u \neq 0} \frac{\|Hu\|}{\|u\|} \end{cases}$$

difficult to calculate

Easier way

$$\boxed{\|E\| = \max_{i,j \in \{1,2,3\}} (|E_{ij}|)}$$

NOT coordinate invariant but OK for saying E is small

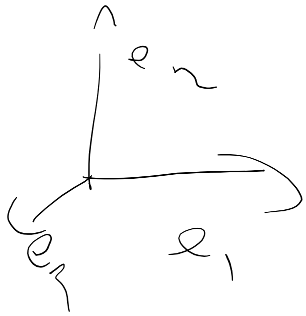
$$\|E\| < \varepsilon \ll 1$$

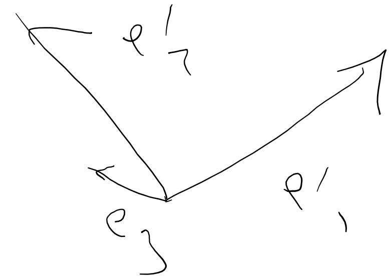
$$F = \begin{bmatrix} 1.001 & -0.03 & 0.01 \\ & 1.008 & 0.02 \\ & & 0.991 \end{bmatrix}$$

$$\Rightarrow H = F - I = \begin{bmatrix} 0.001 & -0.03 & 0.01 \\ & .008 & 0.02 \\ & & 0.009 \end{bmatrix}$$

$\boxed{\|H\| = 0.02} \ll 1$? maybe yes

$H'_{ij} = Q_{im}Q_{jn}H_{mn}$





$$|H'_{ij}| \le \Big|\underbrace{\sum_{m,n} Q_{im}Q_{jn}H_{mn}}_{9\text{ terms}}\Big| \le \sum_{m,n} |Q_{im}Q_{jn}H_{mn}|$$

triangular

$|Q_{im}|, |Q_{jn}| \le 1$

$$\le \sum_{m,n} 1 \cdot 1 \, \underbrace{|H_{mn}|}_{\le \|H\|}$$

$$|H'_{ij}| \le 9\|H\|$$

$$\xRightarrow{i,j} {}_{9}\|H\| \le \|H'\| \le 9\|H\|$$

Similar

if H is small in one system it's small

'If H is small in one system it's small in all systems

Definition 82 *Given a displacement gradient field* $\mathbf{H}$ *derived from a deformation* $\mathbf{f}$ *on* $\overset{0}{\mathcal{B}}$*, a* norm of the displacement gradient field,[8] *denoted* ε*, is*

$$\varepsilon = \|\mathbf{H}\| := \sup_{\substack{\mathbf{x}\in\overset{0}{\mathcal{B}} \\ i,j\in\{1,2,3\}}} |H_{ij}(\mathbf{x})| .$$

Notation

$$f(x+\Delta x) = f(x) + \Delta x f'(x) + \frac{\Delta x^2}{2} f''(x) + \frac{\Delta x^3}{3!} f'''(x) \ldots$$

$$\underbrace{\Delta x^2 \underbrace{\left(\frac{f''(x)}{2} + \Delta x \frac{f'''(x)}{6} + \ldots \right)}_{}}_{\Delta x \to 0}$$

$$= f(x) + \Delta x f'(x) + \underbrace{O(\Delta x^2)}_{g(x)}$$

big O

$$\leq C \Delta x^2$$

as $\Delta x \to 0$

$$\boxed{\begin{gathered} g(x) = O(\Delta x^p) \\ \equiv \\ |g(x)| \leq C \Delta x^p \\ \text{as } \Delta x \to 0 \\ p > 0 \end{gathered}}$$

some constant C

$$G_{ij} = E_{ij} + \frac{1}{2} H_{im} H_{jm}$$

$$|G_{ij} - E_{ij}| = \left|\frac{1}{2} H_{im} H_{jm}\right|$$

Σm

$$= \left|\frac{1}{2} H_{i1} H_{j1} + H_{i2} H_{j2} + H_{i3} H_{j3}\right|$$

$$\leq \frac{1}{2}\Big(\underbrace{|H_{i1} H_{j1}|}_{\leq \varepsilon \cdot \varepsilon} + |H_{i2} H_{j2}| + |H_{i3} H_{j3}| \Big)$$

triangular

$\varepsilon \cdot \varepsilon$ $\varepsilon \cdot \varepsilon$

$$\forall_{I,J}^{(x)} |H_{IJ}(x)| < \varepsilon$$

$$|G_{ij} - \cancel{E}_{ij}| < 3\varepsilon^2$$

$$\boxed{\|G_{ij} - E_{ij}\| < \underbrace{\tfrac{3}{2}}\,\varepsilon^2}$$

$$G_{ij} - E_{ij} = O(\varepsilon^2)$$

$$\boxed{G = \underbrace{E}_{O(\varepsilon)} + O(\varepsilon^2)}$$