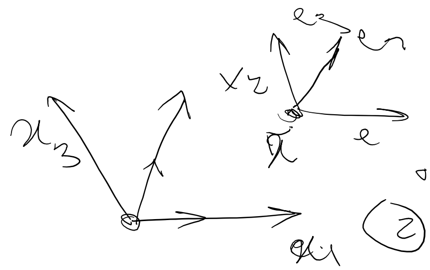
2019/10/14

Monday, October 14, 2019 11:38 AM

Approximating strain equations for infinitesimal theory:



(1) $\varepsilon(x, e_i) = \sqrt{e_i \cdot C e_{\underline{i}}} - 1 = U e_i - 1$

rel. change of length

no summation

(2) $\sin \gamma_{ij} = \dfrac{e_i \cdot C e_j}{\sqrt{e_i \cdot C e_{\underline{i}}}\sqrt{e_j \cdot C e_{\underline{j}}}}$

// change of angle

$\frac{\pi}{2} - \theta_{y_{ij}}$

$$\nabla y = \nabla u + I$$

$$F = H + I$$

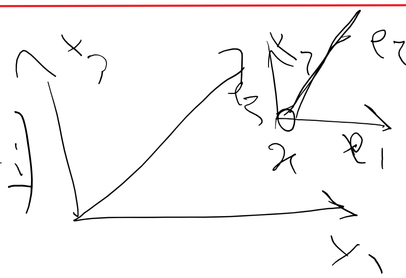
$\varepsilon = \mathrm{Max}\left(|H_{ij}(x)|\right)$

$\forall x \in D$ & $i, j \in \{1,2,3\}$

ε small : infinitesimal theory / approximation

Def. 82

First, we find the approximate form for relative change of length:



$$\varepsilon(x, e_i) = \sqrt{C_{i\underline{i}}} - 1 \quad \left(C_{i\underline{i}} = e_i \cdot C e_{\underline{i}}\right)$$

$$C = I + \underbrace{H + H^T}_{2E} + \underbrace{H^T H}_{O(\varepsilon^2)} = I + 2G \longrightarrow \text{Green strain}$$

$$= \underline{I + 2E + O(\varepsilon^2)}$$

$E = \frac{1}{2}(H + H^T)$

$E = O(\varepsilon)$

$$C_{i\underline{i}} = \left(I + 2E + O(\varepsilon^2)\right)_{i\underline{i}} =$$

$$\delta_{i\underline{i}} + 2E_{i\underline{i}} + O(\varepsilon^2)$$

$$= 1 + 2E_{ii} + O(\varepsilon^2)$$

$\boxed{\begin{aligned} &\sqrt{1+x} = (1+x)^{\frac{1}{2}} \\ &|x| < 1 \quad = 1 + \tfrac{1}{2}x \\ &\qquad + \tfrac{1}{2}\left(\tfrac{1}{2}-1\right)x^2 \end{aligned}}$

$$= 1 + 2E_{i\underline{i}} + O(\varepsilon^2)$$

$|x|<1$ $= 1 + \frac{1}{2}x + \frac{1}{2}\left(\frac{1}{2}-1\right)x^2 + \ldots$

$$\varepsilon(x, e_i) = \sqrt{1 + \underbrace{\underbrace{(2E_{i\underline{i}} + O(\varepsilon^2))}_{O(\varepsilon)}}_{\text{very small} \ll 1}} - 1 =$$

$$\cancel{1} + \frac{1}{2}\left(2E_{i\underline{i}} + O(\varepsilon^2)\right) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(\overbrace{2E_{ii} + O(\varepsilon^2)}^{O(\varepsilon)}\right)^2}{2 \ldots} - \cancel{1}$$

$$= E_{i\underline{i}} + \underbrace{O(\varepsilon^2) + O(\varepsilon^2)}\ldots$$

$$= E_{i\underline{i}} + O(\varepsilon^2)$$

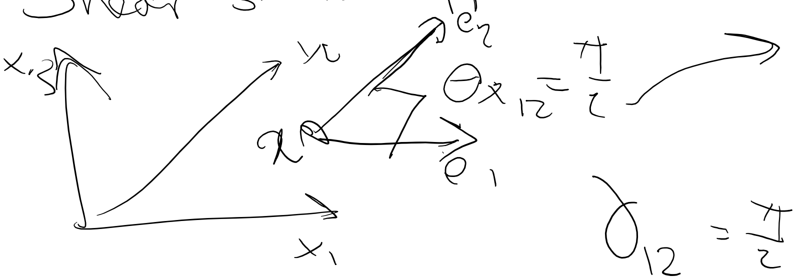
1'

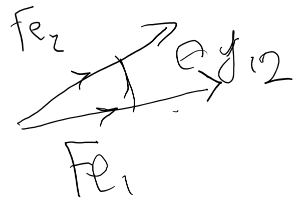
$$\boxed{\varepsilon(x, e_i) = E_{i\underline{i}} + O(\varepsilon^2)}$$

$\varepsilon = \|H\|$ the norm we defined

no summation on $\underline{i}$

Shear strain approximation





$$\gamma_{12} = \frac{\pi}{2} - \theta y_{12}$$

In general $i \neq j$ $\quad \sin\gamma_{ij} = \dfrac{e_i \cdot C e_j}{\sqrt{e_i \cdot C e_i}\,\sqrt{e_j \cdot C e_j}}$

$$C = I + 2E + O(\varepsilon^2)$$

$$e_i \cdot C e_j = C_{ij} = (\underbrace{\delta_{ij}}_{0} + 2E_{ij} + O(\varepsilon^2)) \qquad i \neq j$$

$$= 2E_{ij} + O(\varepsilon^2)$$

$$e_i \cdot C e_i = C_{ii} = 1 + \underbrace{\underbrace{2E_{ii}}_{O(\varepsilon)} + O(\varepsilon^2)}_{O(\varepsilon)} \qquad E = \frac{H + H^T}{2}$$

$$e_j \cdot C e_j = C_{jj} = 1 + O(\varepsilon)$$

$$\sqrt{1 + \underbrace{O(\varepsilon)}_{x}} = 1 + \underbrace{\frac{1}{2}x}_{O(\varepsilon)} - \underbrace{\frac{1}{4}x^2}_{O(\varepsilon^2)} \cdots$$

$O(\varepsilon)$

$$\sin\gamma_{ij} = \underbrace{(2E_{ij} + O(\varepsilon^2))}_{C_{ij}} \overbrace{\underbrace{(1 + O(\varepsilon))}_{\frac{1}{\sqrt{C_{ii}}}} \underbrace{(1 + O(\varepsilon))}_{\frac{1}{\sqrt{C_{jj}}}}}^{= 1 + O(\varepsilon)}$$

larger terms

$$= 2E_{ij} + O(\varepsilon^2)$$

$$\sin\gamma_{ij} = 2E_{ij} + O(\varepsilon^2) = O(\varepsilon)$$

$$\sin\theta_{ij} = 2\underbrace{\epsilon_{ij}}_{O(\epsilon)} + O(\epsilon^3) = O(\epsilon^3)$$

Small number

$$\underbrace{\sin\gamma_{ij}}_{\text{since this is } O(\epsilon)} = \overbrace{\gamma_{ij}}^{O(\epsilon)} + \underbrace{\frac{1}{3!}\gamma_{ij}^3}_{O(\epsilon^3)} \cdots$$

$$\boxed{\gamma_{ij} = 2\underbrace{E_{ij}}_{} + O(\epsilon^2)}$$

2

engineering shear strain — Tensorial strain

"Actual change of angle" between e_i & e_j → half of

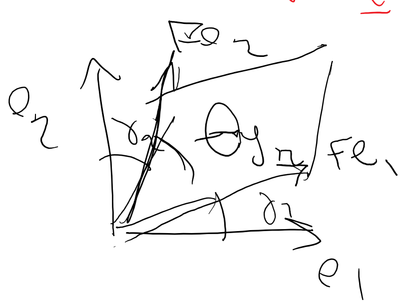
Summary

$$\epsilon(x, e_i) = \sqrt{C_{ii}} - 1 = E_{ii} + O(\epsilon^2)$$

$$\xrightarrow{Fe_i}$$
$$\xrightarrow[e_i]{}$$

$$\epsilon(x, e_1) = \frac{|Fe_i|}{|e_i|} - 1$$

$$\sin\gamma_{ij}(x) = \frac{C_{ij}}{\sqrt{C_{ii}}\sqrt{C_{jj}}} \Longrightarrow \gamma_{ij} = 2E_{ij} + O(\epsilon^2) \quad i \neq j$$



$$\gamma_{xy} = \gamma_1 + \gamma_2 = 2E_{xy}$$

$$E_{xy} = \text{Approximately } (O(\epsilon^2))$$

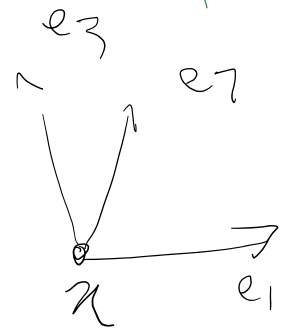
half of change of angle

half of change of angle

$$E = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix}$$

shear strains = half of change of angle

normal strains along e_1, e_2, e_3 directions



$$= \frac{H + H^T}{2}$$

$$= \left(\frac{\nabla u + \nabla u^T}{2} \right) \qquad H = \nabla u$$

$u(x)$ displacement vector $\longrightarrow \nabla u$

matrix

2nd order tensor

So is $\nabla u^T \longrightarrow$

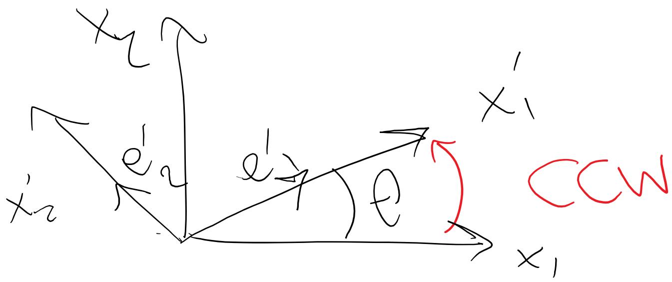
$E = \dfrac{\nabla u + \nabla u^T}{2}$ is a ~~sec~~ second order tensor!

likewise $\quad \omega = \dfrac{\nabla u - \nabla u^T}{2}$ Also a tensor

Now that we know it's a tensor, we know that it follows coordinate transformation rules:

$$E'_{ij} = Q_{im} Q_{jn} E_{mn}$$

2D



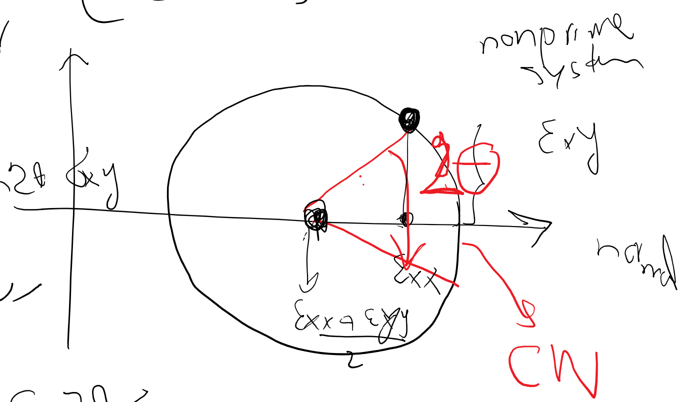
$$Q = \begin{bmatrix} e'_1 \\ \hline e'_2 \end{bmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \qquad c = \cos\theta, \quad s = \sin\theta$$

$\Rightarrow$ Show that

$$\varepsilon_{x'x'} = \left(\frac{\varepsilon_{xx}+\varepsilon_{yy}}{2}\right) + \left(\frac{\varepsilon_{xx}-\varepsilon_{yy}}{2}\right)\cos 2\theta - \sin 2\theta \, \varepsilon_{xy}$$

$$\varepsilon_{y'y'} = \quad '' \quad - \quad '' \quad + \quad ''$$

$$\varepsilon_{x'y'} = \left(\frac{\varepsilon_{xx}-\varepsilon_{yy}}{2}\right)\sin 2\theta + \cos 2\theta \, \varepsilon_{xy}$$



E is a tensor, but components of normal and shear strains do not transfer like tensor for large deformation theory

2D $\quad E = \begin{bmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{bmatrix}$

3D $\quad E = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix}$

redundant & not included in Voigt notation

$$\gamma = \begin{bmatrix} E_{11} \\ E_{22} \\ 2E_{12} \end{bmatrix} \quad \gamma_{12}$$

normal

$$\gamma = \begin{Bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{12} \\ 2E_{23} \\ 2E_{31} \end{Bmatrix}$$

γ_{12}, γ_{23}, γ_{31} $\leftarrow$ shear components

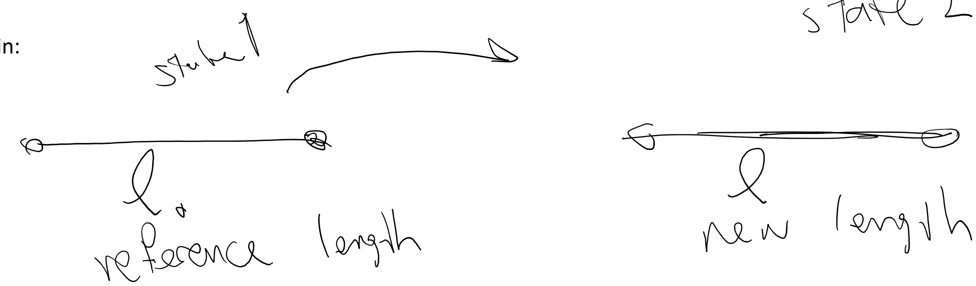
engineering shear strains

Voigt notation

1-array (not a vector)

A more general definition of strain:

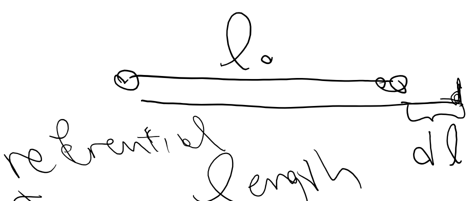
1D



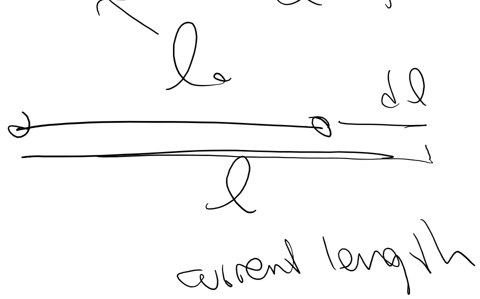
$$\varepsilon = \frac{\Delta l}{l_0} = \frac{l - l_0}{l_0} \qquad \Delta l = l - l_0$$

How about this

$$\varepsilon = \int d\varepsilon = \int_{l_0}^{l} \frac{dl}{l_0} = \frac{1}{l_0}\int_{l_0}^{l} dl = \frac{\Delta l}{l_0}$$



$$d\varepsilon = \frac{\text{change of length}}{\text{original length}}$$



$$d\varepsilon = \frac{dl}{\cancel{l_0}} = \frac{dl}{l}$$

current length

$$\varepsilon = \int_{l_0}^{l} \frac{dl}{l} = \mathrm{Ln}\left(\frac{l}{l_0}\right)$$

logarithmic strain

$$\varepsilon = \frac{\Delta l}{l_0}$$

change of length normalized by initial length

$$\varepsilon = \mathrm{Ln}\left(\frac{l}{l_0}\right) = \mathrm{Ln}\left(\frac{\Delta l}{l_0} + 1\right) \approx \frac{\Delta l}{l} + O\left(\left(\frac{\Delta l}{l}\right)^2\right)$$

$$\varepsilon = \ln\left(\frac{l}{l_0}\right) = \ln\left(\frac{\Delta l}{l_0}+1\right) \simeq \frac{\Delta l}{l_0} + O\left(\left(\frac{\Delta l}{l_0}\right)^2\right)$$

at every stage of deformation

increment of strain is computed using the current length

Example

apply force

l_0 $\quad F$

$\frac{1}{2} l_0$

$\varepsilon_{lin} = \frac{\Delta l}{l_0} = \frac{-\frac{1}{2} l_0}{l_0} = -\frac{1}{2}$

$\varepsilon_{log} = \ln\left(\frac{\frac{1}{2} l_0}{l_0}\right) = \ln\left(\frac{1}{2}\right)$

$\frac{1}{4} l_0 \rightarrow \varepsilon_l = -\frac{3}{4}$

$\varepsilon_{log} = \ln\left(\frac{1}{4}\right)$

$F = k\varepsilon$

for ε_{log} $F = k\varepsilon$ is OK as $l \to 0$

$l \to 0$

$\varepsilon_l = \frac{\Delta l}{l_0}$ as $l \to 0$ $= -1$

But physically

$F = k\varepsilon$

for ε_{lin} is a very good model (need to use a better $\varepsilon \to F$ const. eqn)

$\varepsilon_{log} = \log\left(\frac{l}{l_0}\right) = -\infty$ $\quad l \to 0$

Other strain definitions:

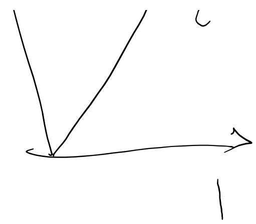
strain = stretch − 1

rigid motion: $\quad 0 \qquad 1$

~ 3

$\varepsilon = \sqrt{C} \ldots$

$$\underbrace{\varepsilon_i}_{\text{strain}} = \sqrt{C_{ii}} - 1 = \underbrace{U_{ii}}_{\text{stretch}} - 1$$



$$= \sqrt{\underbrace{1 + 2G_{ii}}_{C_{ii}}} - 1 \approx 1 + \frac{1}{2}\,2G_{ii} - 1 = G_{ii}$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x + O(x^2)$$

$$\simeq G_{ii}$$

$$G = \frac{1}{2}(C - 1)$$
$$= \frac{1}{2}(U^2 - 1)$$

$U - 1$ Exact strain

$G = \frac{1}{2}(U^2 - 1)$ approximate strain

$\ln U$ logarithmic strain

The various measures of Lagrangian strain used in the literature are all related to the stretch U in a one-to-one manner. Examples include,

$U - I$ (exact) linear strain

$G = \frac{1}{2}(U^2 - I)$ Green strain

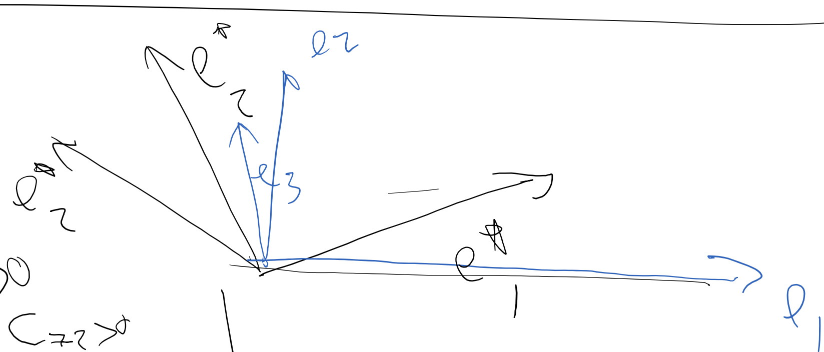
$$\frac{1}{m}(U^m - I)$$ generalized Green strain

m: nonzero integer

$$\ln U$$ Logarithim strain

(Hencky strain)

Define $\ln U$:



$$C = F^T F$$

$$[C]^* = \begin{bmatrix} C_{11}>0 & & \\ & C_{22}>0 & \\ & & C_{33}>0 \end{bmatrix}$$

principal axes of C

$$U = \sqrt{C} \qquad [U]^* = \begin{bmatrix} \underbrace{\sqrt{C_{11}}}_{U_{11}} & & \\ & \underbrace{\sqrt{C_{22}}}_{U_{22}} & \\ & & \overbrace{\sqrt{C_{33}}}^{U_{33}} \end{bmatrix}$$

$$[\ln U]^* = \begin{bmatrix} \ln U_{11} & 0 & 0 \\ 0 & \ln U_{22} & 0 \\ 0 & 0 & \ln U_{33} \end{bmatrix}$$

Other definitions of strain

$e(\lambda)$ λ eigenvalues of U

$$e(\lambda): \qquad [\text{strain}]^* = \begin{bmatrix} e(U_{11}) & & \\ & e(U_{22}) & \\ & & e(U_{33}) \end{bmatrix}$$

$e(U_{33})$

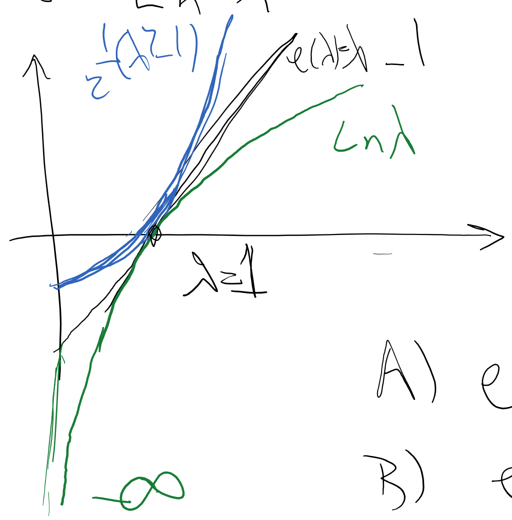
Examples

$e(\lambda) = \lambda - 1 \longrightarrow U - 1$

$e(\lambda) = \frac{1}{2}(\lambda^2 - 1) \longrightarrow \frac{1}{2}(U^2 - 1) = G$

$e(\lambda) = \frac{1}{m}(\lambda^m - 1) \longrightarrow \frac{1}{m}(U^m - 1)$

$e(\lambda) = \operatorname{Ln} \lambda \longrightarrow \underbrace{\operatorname{Ln} U}_{\text{Corresponding strain tensors}}$



A) $e(\lambda = 1) = 0$

B) $e'(\lambda = 1) = 1$

C) all monotonically increasing

$\Vert$ $e'(\lambda) > 0$ for all $\lambda > 0$

$$\varepsilon_\lambda = E + O(\varepsilon^2)$$

next time