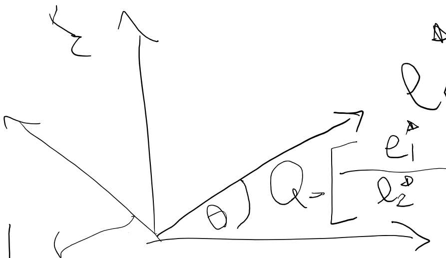
2019/10/16

Wednesday, October 16, 2019 11:40 AM

Different definitions of strain tensor

$C = F^t F \longrightarrow U = \sqrt{C}$



$Q = \begin{bmatrix} e_1^* \\ e_2^* \end{bmatrix}$, $Q = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$, $c = \cos\theta$, $s = \sin\theta$

$[C]^* = \begin{bmatrix} C_{11}^* & 0 & 0 \\ 0 & C_{22}^* & 0 \\ 0 & 0 & C_{33}^* \end{bmatrix}$

$[U]^* = \begin{bmatrix} \sqrt{C_{11}^*} & 0 & 0 \\ 0 & \sqrt{C_{22}^*} & 0 \\ 0 & 0 & \sqrt{C_{33}^*} \end{bmatrix} = \begin{bmatrix} U_{11}^* & 0 & 0 \\ 0 & U_{22}^* & 0 \\ 0 & 0 & U_{33}^* \end{bmatrix}$

General strain $[e(U)]^* = \begin{bmatrix} e(U_{11}^*) & 0 & 0 \\ 0 & e(U_{22}^*) & 0 \\ 0 & 0 & e(U_{33}^*) \end{bmatrix}$ (I) $\longrightarrow$ generalized normal strain

$[U]$, $U_{ij}^* = Q_{im} Q_{jn} U_{mn} \Rightarrow U^* = Q U Q^t$

xy system

$U = Q^t U^* Q$

$$[e(U)] = Q^t \underbrace{[e(U)^*]}_{\text{eqn (1)}} Q$$

Three conditions we had for e function:

1) $e(1) = 0$

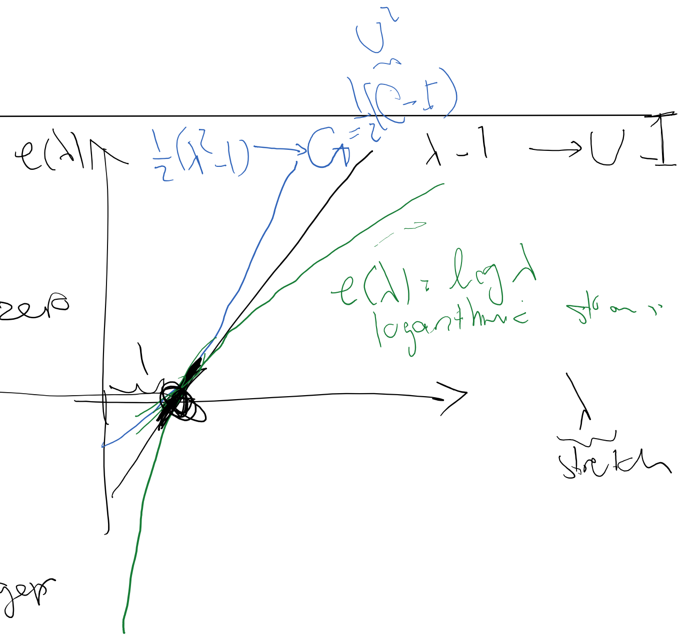
for stretch $= 1 \to$ strain is zero

2) $e'(1) = 1$

3) $e'(\lambda) > 0$ for $\lambda > 0$

$\lambda \nearrow \Rightarrow$ strain got larger

higher stretch $\Longleftrightarrow$ higher sense



$$\boxed{[E_e]^* = \begin{bmatrix} e(U_{11}^*) & & \\ & e(U_{22}^*) & \\ & & e(U_{33}^*) \end{bmatrix} \quad \text{(I)}}$$

3 $\quad * \quad * \quad *$

$$\boxed{E_e = \sum_{i=1}^{3} e(U_{ii}^{*})\, e_i^{*} \otimes e_i^{*}}$$

Claim. All these strains are equal to within ε^2

$$\boxed{\varepsilon = \max_{i,j,\, x \in D} H_{ij}(x)}$$

$$\underbrace{e(\lambda)}_{\text{stretch}} = e(\lambda - 1 + 1) = \underbrace{e(1)}_{0} + (\lambda-1)\underbrace{e'(1)}_{=1 \;\to 2)} + \frac{(\lambda-1)^2}{2} e''(1) \cdots$$

$$e(\lambda) = \lambda - 1 + \underbrace{\frac{(\lambda-1)^2}{2} e''(1) + H.O.T.}$$

$$[E_e]^{*} = \begin{bmatrix} U_{11}^{*}-1 & & \\ & U_{22}^{*}-1 & \\ & & U_{33}^{*}-1 \end{bmatrix} + \underline{O(\varepsilon^2)}$$

we can show this

$$\begin{cases} U_{11}^{*} - 1 = O(\varepsilon) \\ U_{22}^{*} - 1 = O(\varepsilon) \\ U_{33}^{*} - 1 = O(\varepsilon) \end{cases}$$

$$\boxed{\begin{aligned} E_e &= \underbrace{(U - I)}_{\text{Exact stretch}} + O(\varepsilon^2) \\ &= \frac{1}{2}\underbrace{(H + H^T)} + O(\varepsilon^2) \\ &= \underbrace{E}_{\text{Infinitesimal strain}} + O(\varepsilon^2) \end{aligned}}$$

E_e, $\underbrace{U - 1}_{\text{exact}}$, E

are to within ε^2

$$\begin{cases} e_\lambda = \frac{1}{2}(\lambda^2 - 1) \Rightarrow E_e = \frac{1}{2}(U^2 - 1) \\ \qquad\qquad = \frac{1}{2}(\underbrace{C}_{\text{easy to calculate as } C \text{ is needed not } U = \sqrt{C}} - 1) = E_G \\ e_\lambda = \log \lambda \Rightarrow \text{logarithmic strain} \end{cases}$$

and $U = \sqrt{C}$

Summary:

Finite deformation	Infinitesimal deformation
full rotation	infinitesimal rotation
Rigid: $y = Qx + c$	$u = \underset{\sim}{W}x + c$ skew sym.
Deformation: $y - x = \underbrace{(Q - I)}x + c$	approximated by W
$C = I, U = I, G = 0$	$E = 0$ $\quad = G = O(\varepsilon^2)$
$E = O(\varepsilon^2)$	

General deformation

$F = RU = VR$

∇y_x

rotation

right & left (Lagrangian U, Eulerian V) stretches

$H = \underbrace{\left\{\frac{1}{2}(H + H^T)\right\}}_{\varepsilon} + \underbrace{\left\{\frac{1}{2}(H - H^T)\right\}}_{W}$

∇u_x

$\varepsilon + W$

infin. strain, infin. rotation

$F = F^p F^e$ large def. plasticity

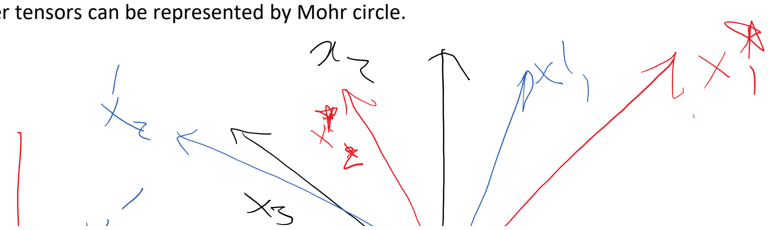
$\varepsilon = \varepsilon_{elastic} + \varepsilon_{plastic}$

Mohr circle:

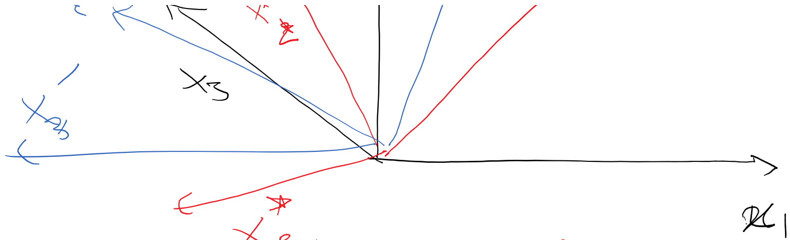
Coordinate transformation rule for symmetric second order tensors can be represented by Mohr circle.

E

$[E^*] = [E_{11}^*$ ... E^*



$$[E]^{*} = \begin{bmatrix} E^{*}_{11} & & \\ & E^{*}_{22} & \\ & & E^{*}_{33} \end{bmatrix}$$



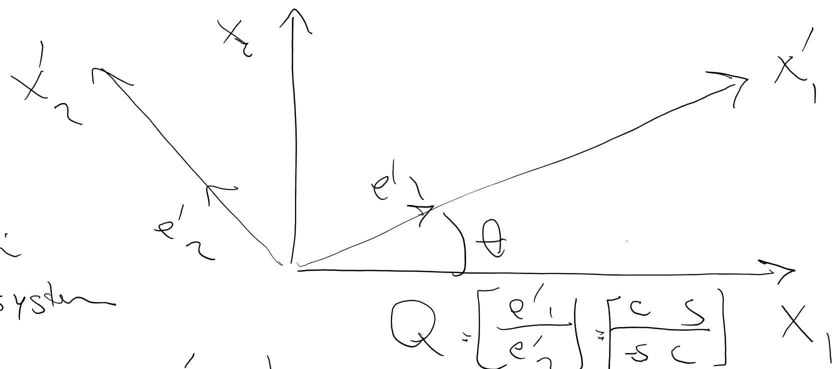
$x_3^* \longrightarrow$ principal axes of sym. E

Mohr circle $\longrightarrow$ $E^{*}_{11}, E^{*}_{22}, E^{*}_{33} = ?$ & what are their directions?

Define components of E in $[\;]'$ system.

3D more complex

2D in class



$$E = \begin{bmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{bmatrix}$$ is given in x_1, x_2 system

$$Q = \begin{bmatrix} e'_1 \\ e'_2 \end{bmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$$

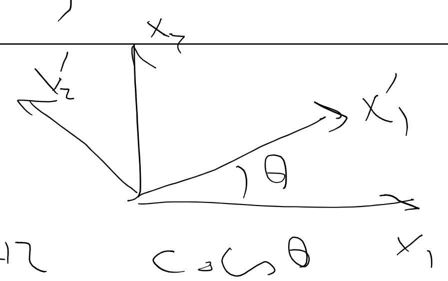
$c = \cos\theta$, $s = \sin\theta$

what $$[E]' = \begin{bmatrix} E'_{11} & E'_{12} \\ E'_{12} & E'_{22} \end{bmatrix}$$ in $(\;)'$ system (x'_1, x'_2)

$$E'_{ij} = Q_{im} Q_{jn} E_{mn} \Longrightarrow [E]' = Q[E]Q^{t}$$

$$\begin{bmatrix} E'_{11} & E'_{12} \\ E'_{12} & E'_{22} \end{bmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{bmatrix} \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$$

$$\Longrightarrow \boxed{\begin{aligned} E'_{11} &= c^2 E_{11} + s^2 E_{22} + 2cs E_{12} \\ E'_{22} &= s^2 E_{11} + c^2 E_{22} - 2cs E_{12} \\ E'_{12} &= -cs(E_{11} - E_{22}) + (c^2 - s^2) E_{12} \end{aligned}} \quad (2)$$



$c = \cos\theta$
$s = \sin\theta$

We can write it as

$$c^2 = \cos^2\theta = \frac{\cos 2\theta + 1}{2}$$

$$s^2 = \sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

$$2cs = 2\sin\theta\cos\theta = \sin 2\theta$$

$$C = \cos(2\theta)$$

$$S = \sin(2\theta)$$

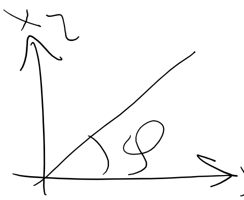
$S = \sin(2\theta)$

normal

shear

$$\begin{bmatrix} E'_{11} \\ E'_{12} \end{bmatrix} = \begin{bmatrix} \frac{E_{11}+E_{22}}{2} \\ 0 \end{bmatrix} + \underbrace{\begin{bmatrix} C & S \\ -S & C \end{bmatrix} \begin{bmatrix} \frac{E_{11}-E_{22}}{2} \\ E_{12} \end{bmatrix}}_{\text{Rotation of angle } 2\theta}$$

(3)



$$R(\varphi) = \left[Re_1 \,\middle|\, Re_2 \right] = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

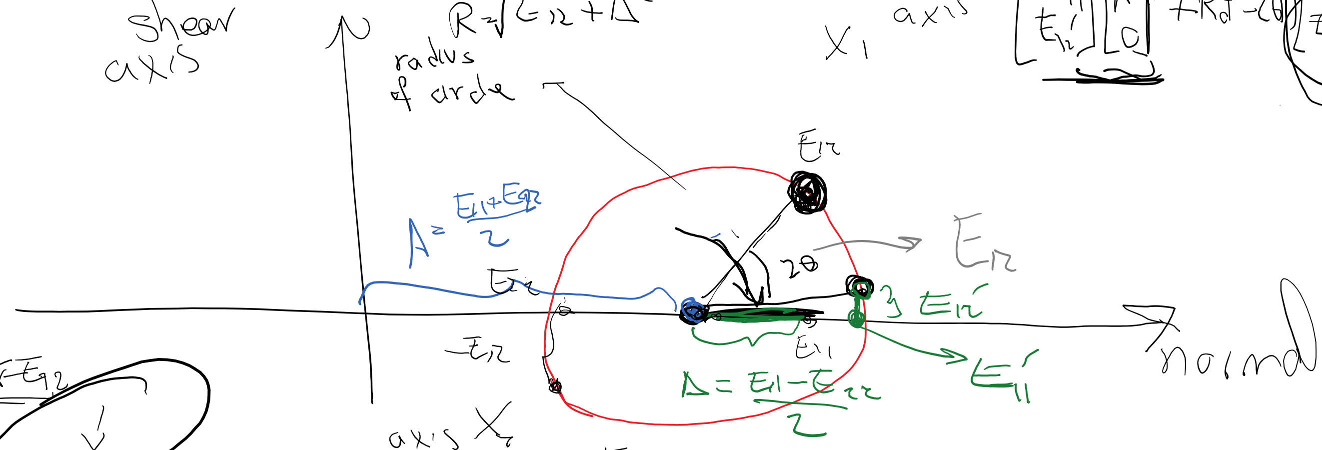
rotation of angle φ

Geometric representation of Mohr circle:

$R = \sqrt{E_{12}^2 + \Delta^2}$

X_1 axis

$$\begin{bmatrix} E'_{11} \\ E'_{12} \end{bmatrix} = \begin{bmatrix} A \\ 0 \end{bmatrix} + R_{-2\theta} \begin{bmatrix} \Delta \\ E_{12} \end{bmatrix}$$

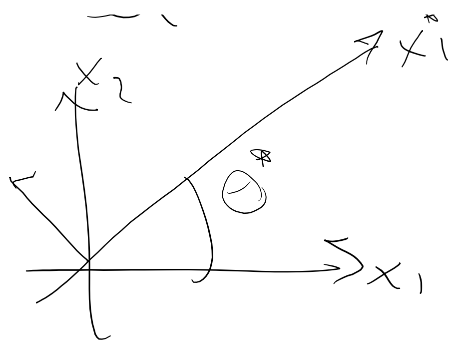


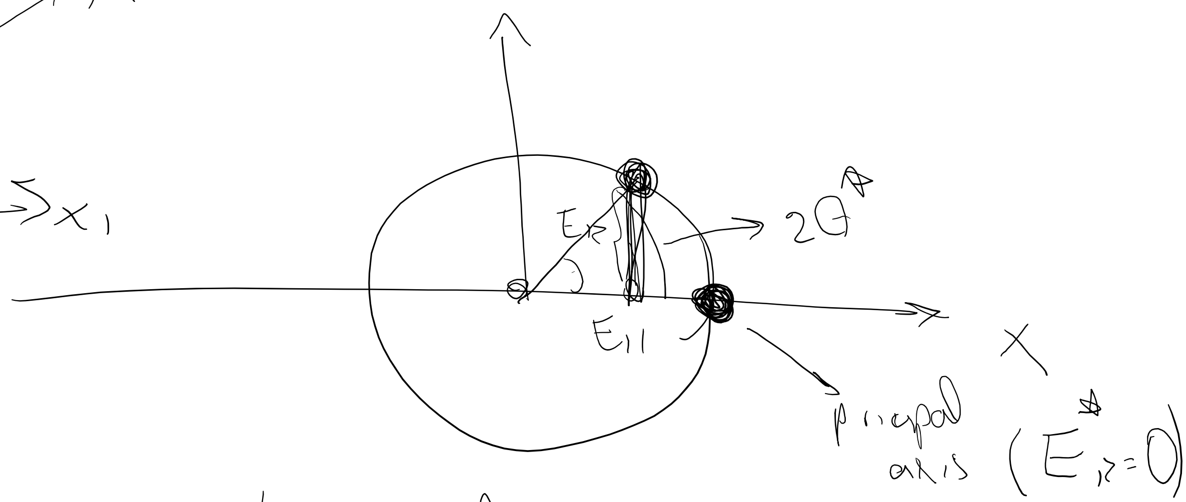
$X_1 \rightarrow$ normal strain E_{11}

shear E_{12}

+ for shear

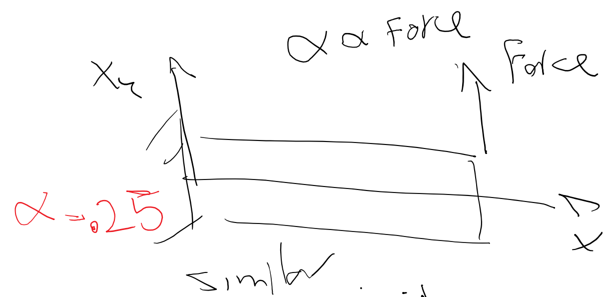
Q: What directions are principal axes





$$\tan 2\theta^* = \frac{E_{12}}{\Delta} \qquad \left(\Delta = \frac{E_{11}-E_{22}}{2}\right)$$

Example:
Computing U,R,V,C,B, different strain tensors, etc.



similar to this problem

$$y_1 = x_1 + \alpha(1-x_1)x_2$$
$$y_2 = \left[1 - \frac{\alpha^2}{2}(1-x_1)^2\right]x_2$$

$$F = \nabla y_x = \begin{bmatrix} y_{1,1} & y_{1,2} \\ y_{2,1} & y_{2,2} \end{bmatrix}$$

$$F = \begin{bmatrix} 1-\alpha x_2 & \alpha(1-x_1) \\ \alpha^2(1-x_1)x_2 & 1-\frac{\alpha^2}{2}(1-x_1)^2 \end{bmatrix}$$

$x_1 = .5$
$x_2 = .2$

$$F = \begin{bmatrix} 0.95 & .125 \\ 0.063 & .992 \end{bmatrix} \qquad H = F - I = \begin{bmatrix} -.05 & .125 \\ 0.0063 & -.0078 \end{bmatrix}$$

$\varepsilon = \mathrm{Max}|H_{ij}| = .125$

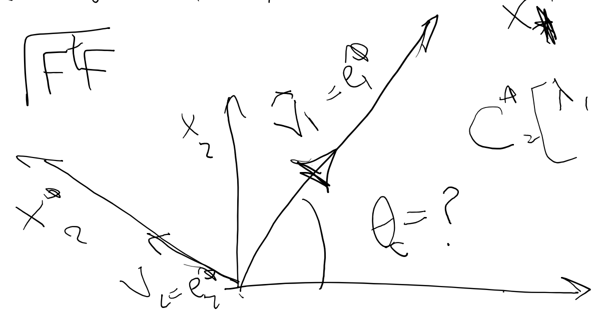
$\varepsilon = \text{Max}|\nabla_{ij}| = .125$

Maybe too large

Part A1 : Large deformation (Right path)

$F = RU \qquad C = F^tF \qquad U = \sqrt{F^tF}$

$$C = F^tF = \begin{bmatrix} 0.9025 & 0.125 \\ .125 & 1.0001 \end{bmatrix}$$



$C^* = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}$

$$C\left[\vec{v}_1 \mid \vec{v}_2\right] = \left[\lambda_1 v_1 \mid \lambda_2 v_2\right] = \underbrace{\left[v_1 \mid v_2\right]}_{\text{eigenvectors}} \underbrace{\begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}}_{\text{eigenvalue}}$$

$\bar{C}\bar{V} = \bar{V}\bar{\Lambda} \Rightarrow$

$\boxed{C = \bar{V}\bar{\Lambda}\bar{V}^{-1}}$ general

C is symmetric we can choose v_i's to be orthonormal

$\Downarrow$

$V^{-1} = V^t$

symmetric C

what I need $\boxed{C = \bar{V}\bar{\Lambda}\bar{V}^T}$

$\bar{V}$ orthonormal

symmetric C

What are the interpretation of $\bar{\Lambda}$ & $\bar{V}$

$\bar{\Lambda} = V^t C V$

$\Downarrow$

$[C]^* = QCQ^t \qquad Q = \begin{bmatrix} e_1^{*t} \\ e_2^{*t} \end{bmatrix}$

$= \bar{V}^t = [v_1 | v_2]^t = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$C_{11}^* = \lambda_1$

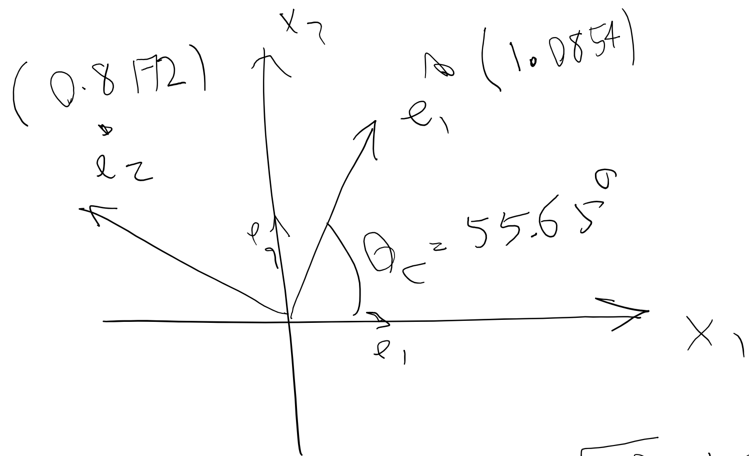
$\underline{\quad V \quad V^t \quad}\rightarrow$

Solve the eigenvalue problem for C

$$\text{eig}(C) = \left\{ (1.0854,\ 0.8172) \left[\underbrace{\begin{bmatrix} .5641 \\ .8257 \end{bmatrix}}_{V_1} \ \underbrace{\begin{bmatrix} -.8257 \\ .5641 \end{bmatrix}}_{V_2} \right] \right\}$$

$$e_1^* = V_1 = \begin{bmatrix} \cos\theta_c \\ \sin\theta_c \end{bmatrix} = \begin{bmatrix} .5641 \\ .8257 \end{bmatrix} \longrightarrow \text{atan2}(.8257, .5641) = 55.65°$$

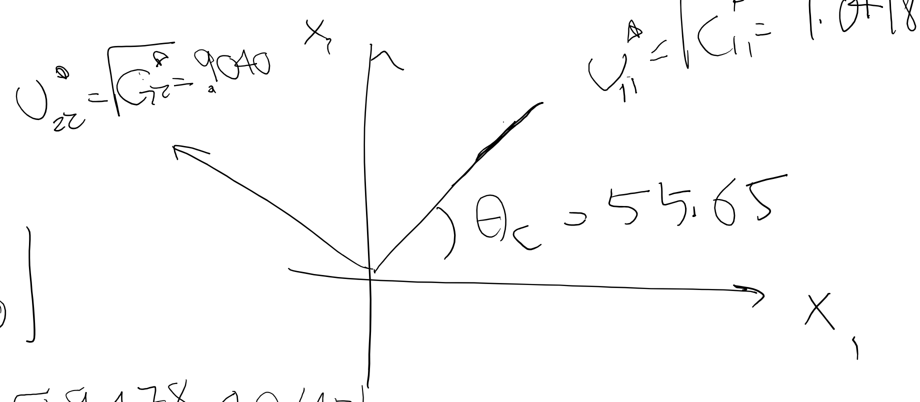
$$[C]^* = \begin{bmatrix} 1.0854 & 0 \\ 0 & 0.8172 \end{bmatrix}$$



$$\boxed{C = V C^* V^t}$$

e_1, e_2 system

$U = ?$

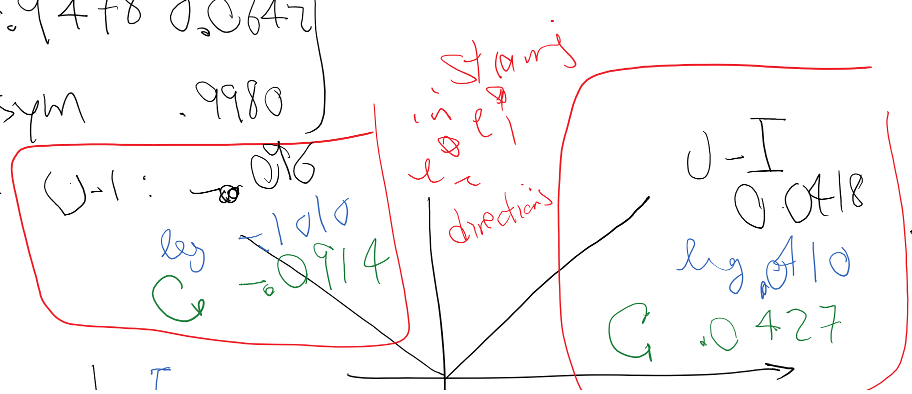


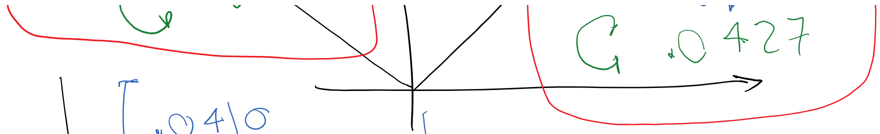
$$[U]^* = \begin{bmatrix} 1.0418 & 0 \\ 0 & 0.9040 \end{bmatrix}$$

$$\underset{\sim xy}{U} = V [U]^* V^t = \begin{bmatrix} 0.9478 & 0.0642 \\ \text{sym} & .9980 \end{bmatrix}$$

3 strains

$U - 1$:





$U - 1:$

$$\log U = \begin{bmatrix} \log(U_{11}) & \\ & \log(U_{22}) \end{bmatrix} = \begin{bmatrix} .0410 & \\ & -.1010 \end{bmatrix}$$

$$G = \frac{U-1}{\varepsilon} = \frac{U-1}{2} = \begin{bmatrix} .0427 & \\ & -.0914 \end{bmatrix}$$