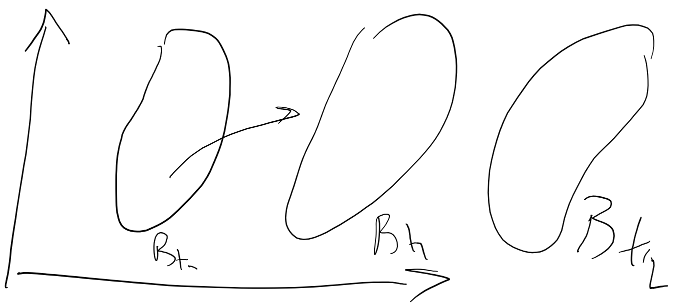
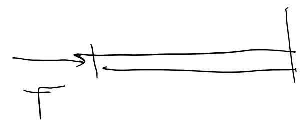
2019/10/23

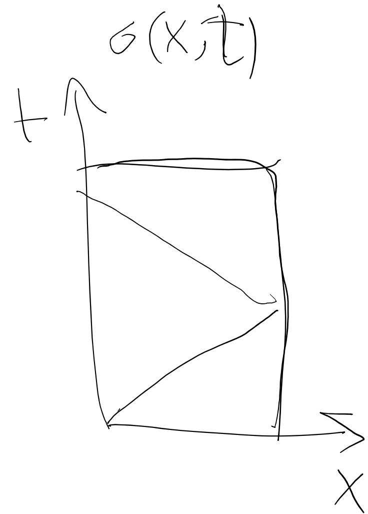
Wednesday, October 23, 2019 11:26 AM

Definition 89: Trajectory of a body under motion f(.,t) is the set of ordered pairs

$$\mathcal{T} = \{(y,t) \mid y \in B_t,\ t \in [t_0,\infty)\}$$







Definitions of velocity and acceleration (Def 90)

$$v(x,t) = \frac{Df(x,t)}{Dt}\Big|_{x\text{-fixed}}$$

$\frac{D}{Dt}$: time derivative when x is fixed

$$a(x,t) = \frac{D\,v(x,t)}{Dt} = \frac{D^2 f}{Dt^2}$$

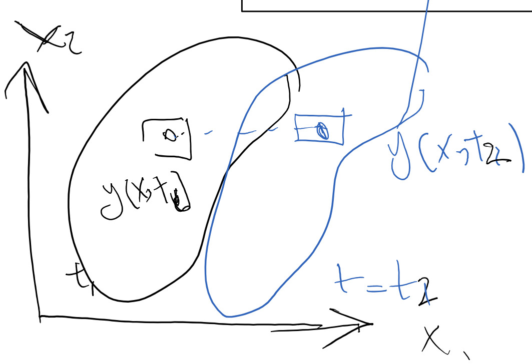
$$y(x,t) = x + u(x,t)$$

$\downarrow$

$f(x,t)$

$$v_i = \frac{Dy_i}{Dt} = \underbrace{\frac{Dx_i}{Dt}\Big|_{x\text{-fixed}}}_{\text{zero}} + \frac{Du_i}{Dt}$$

$$\boxed{\begin{aligned} v(x,t) &= \frac{Du(x,t)}{Dt} = \frac{Dy(x,t)}{Dt} \\ a(x,t) &= \frac{D^2 u(x,t)}{Dt^2} = \frac{D^2 y(x,t)}{Dt^2} \end{aligned}}$$

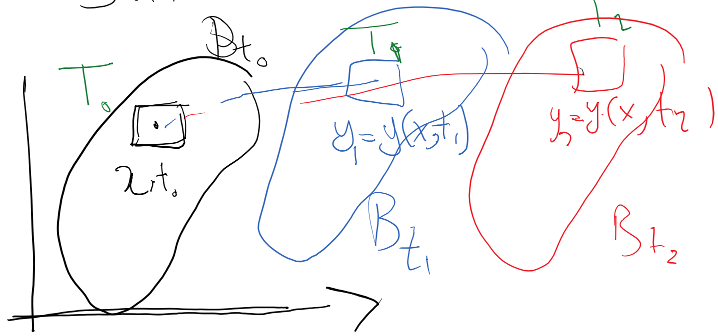


$$v = \lim \frac{y(x,t_2) - y(x,t_1)}{t_2 - t_1}$$

$$t_2 - t_1 \to 0$$

x_1

2 - Different ways of defining rate (time derivative)

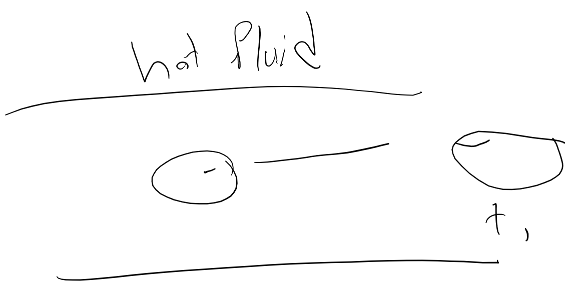


$$\left.\frac{DT}{Dt}\right|_{t_1} = \frac{T(y(x,t_2)) - T(y(x,t_1))}{t_2 - t_1}$$

$t_2 - t_1 \to 0$

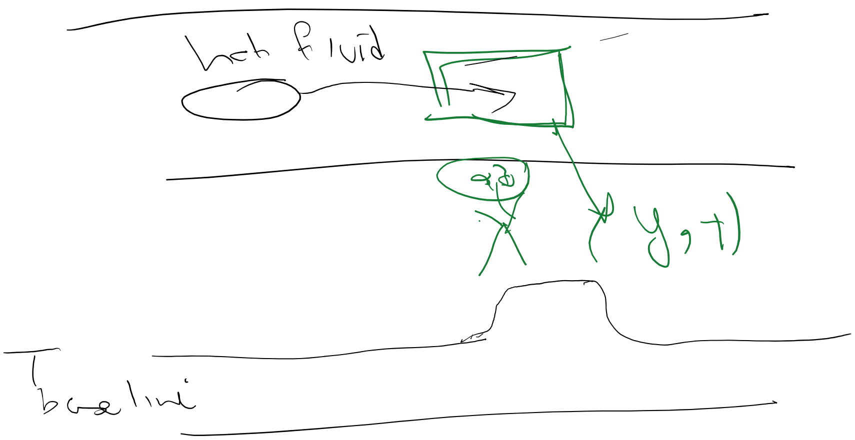
$$= \left.\frac{DT(x,t)}{Dt}\right|_{t_1}$$

Lagrangian time derivative



$\frac{DT}{Dt}$ small

diffusion



$$\left.\frac{\partial T}{\partial t}\right|_{y\text{-fixed}}$$

fixed observation window

we're not following the particle

Eulerian time derivative

Relation between

$$\left.\frac{\partial T}{\partial t}\right|_{y\text{-fixed}} \qquad \left.\frac{DT}{Dt}\right|_{x\text{-fixed}} \qquad (\nabla T)_i \, V_i = \nabla_y T \, V$$

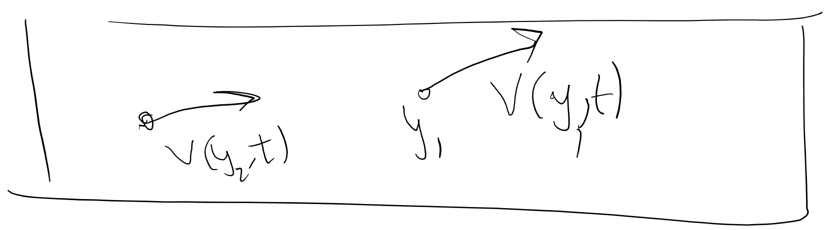
$$DT(y,t) \qquad \partial T(y,t), \quad \left(\frac{\partial T}{\partial y_i}\right)\left(\frac{\partial y_i}{\partial t}\right)$$

$$\frac{DT(y,t)}{Dt}\bigg|_{x\text{-fixed}} = \frac{\partial T(y,t)}{\partial t} + \left(\frac{\partial T}{\partial y_i}\right)\left(\frac{\partial y_i}{\partial t}\right)\bigg|_{x\text{-fixed}}$$

$$\boxed{\underbrace{\frac{DT(y,t)}{Dt}\bigg|_{x\text{-fixed}}}_{\text{Lagrangian rate}} = \underbrace{\frac{\partial T(y,t)}{\partial t}\bigg|_{y\text{-fixed}}}_{\text{Eulerian.}} + \underbrace{\left(\nabla_y T\right)}_{\text{grad}\,T} v}$$

Example $T = V$

$a = ?$



$$a = \cancel{\frac{\partial V(y,t)}{\partial t}}$$

we are not following material

$$\boxed{a = \frac{D\hat{V}}{Dt} = \frac{\partial \hat{V}}{\partial t} + \left(\nabla_y \hat{V}\right)\hat{V}}$$

in course notes $\hat{T}$ is representation of T as a function of y,t — Eulerian representation

$\underline{T}$ is its representation in (x,t)

Lagrangian representation

$$\frac{DT}{Dt}\bigg|_{x\text{-fixed}} \longleftrightarrow \frac{\partial \hat{T}}{\partial t}\bigg|_{y\text{-fixed}}$$

$$\operatorname{Grad} T(x,t)\big|_{x\text{-fixed}} \qquad \operatorname{grad}\hat{T}\big|_{y\text{-fixed}}$$

$$\nabla_x T \qquad \nabla_y \hat{T}$$

$$\operatorname{Div} T(x,t) \quad \text{HW} \quad \operatorname{div}\hat{T}(y,t)$$

$$= \operatorname{trace}(\operatorname{Grad} T) \qquad = \operatorname{trace}(\operatorname{grad}\hat{T})$$

Consider vector ω ($\vec{\omega}(x,t)$, $\hat{\omega}(y,t)$)

$$(\operatorname{Grad}\hat{\omega})_{ij} = \frac{\partial \hat{\omega}_i}{\partial X_j}\bigg|_{x\text{-fixed}} = \frac{\partial \hat{\omega}_i(y,t)}{\partial X_j} =$$

$$\underbrace{\frac{\partial \hat{\omega}_i(y,t)}{\partial y_k}}_{(\operatorname{grad}\hat{\omega})_{ik}} \underbrace{\frac{\partial y_k}{\partial X_j}}_{F_{kj}} = (\operatorname{grad}\hat{\omega})_{ik} F_{kj}$$

$$\operatorname{Grad}\hat{\omega} = \operatorname{grad}\omega \cdot F \Longleftrightarrow \operatorname{grad}\omega = \operatorname{Grad}\omega \cdot F^{-1}$$

This holds true for any order tensor

$$(\operatorname{Grad}\hat{T}(y,t))_{i_1 \cdots i_m j} = \frac{\partial \hat{T}_{i_1 \cdots i_m}}{\partial X_j} = \underbrace{\frac{\partial \hat{T}_{i_1 \cdots i_m}}{\partial y_k}}_{\operatorname{grad}\hat{T}_{i_1 \cdots i_m k}} \underbrace{\frac{\partial y_k}{\partial X_j}}_{F_{kj}}$$

grad $\tilde{T}_{ij\ldots ink}$ f_{kj}

Same relation

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rate of J

$$J = \det F$$

$$\frac{DJ}{Dt} = ?$$

$$J = \frac{dV_y}{dV_x} = \frac{\text{new volume}}{\text{old volume}}$$

$$\varepsilon_v = J - 1$$

Volumetric strain

$$\approx E_{11} + E_{22} + E_{33} + O(\varepsilon^2)$$

HW

$$\frac{DJ}{Dt} = \frac{D \det F}{Dt}$$

material rate of J:

Side note: $\frac{d(\det A)}{d\alpha} = \text{trace}\left(\frac{dA}{d\alpha} A^{-1}\right) \det A$

$$\frac{DJ}{Dt} = \frac{D \det F}{Dt} = \text{trace}\left(\frac{DF}{Dt} F^{-1}\right) \underbrace{\det F}_{J} \qquad (1)$$

$$\left(\frac{DF}{Dt}\right)_{ij} = \frac{\partial}{\partial t}\left(\frac{\partial y_i}{\partial x_j}\right)\Bigg|_{x\text{-fixed}} = \frac{\partial}{\partial x_j} \underbrace{\left(\frac{\partial y_i}{\partial t}\right)\Bigg|_{x\text{-fixed}}} = \frac{\partial}{\partial x_j} v_i$$

$$= (\underbrace{\text{Grad } v})_{ij} \quad \text{or} \quad \cancel{(\underbrace{\text{grad } v})_{ij}}$$

$$= \underbrace{(\mathrm{Grad}\, V)_{ij}}_{X\text{-coordinate}} \quad \text{or} \quad \underbrace{(\mathrm{grad}\, v)_{ij}}_{y\text{-coordinate}}$$

$$\frac{DF}{Dt} = \mathrm{Grad}\, V \qquad\qquad \mathrm{Grad}\, T = (\mathrm{grad}\, T) F$$

$$\frac{DF}{Dt} F^{-1} = \underbrace{(\mathrm{Grad}\, V) F^{-1}}_{} = \mathrm{grad}\, \hat{v}$$

$$(1) \quad \frac{DJ}{Dt} = \mathrm{trace} \underbrace{\left(\frac{DF}{Dt} F^{-1} \right)}_{} \det F =$$

$$\underbrace{\mathrm{trace}\left(\mathrm{grad}\, \hat{v}\right)}_{} J =$$

$$\mathrm{div}\, \hat{v}\; J$$

Summary

Lagrangian

Eulerian

$$V(X,t) = \frac{Dy(X,t)}{Dt} = \frac{Du(X,t)}{Dt} \qquad \hat{v}(y,t)$$

$$a(X,t) = \frac{DV(X,t)}{Dt} = \frac{D^2 y(X,t)}{Dt^2} = \frac{D^2 u(X,t)}{D^2 t} \qquad \hat{a}(y,t) = \frac{\partial \hat{v}}{\partial t} + \mathrm{grad}\,\hat{v} \cdot \hat{v}$$

$$\frac{DT}{Dt} = \frac{\partial \hat{T}}{\partial t} + \mathrm{grad}\, T\, \hat{v}$$

$$\nabla_X T = \mathrm{Grad}\, T = \underset{\nabla_y \hat{T}}{\mathrm{grad}\, \hat{T}} \cdot F$$

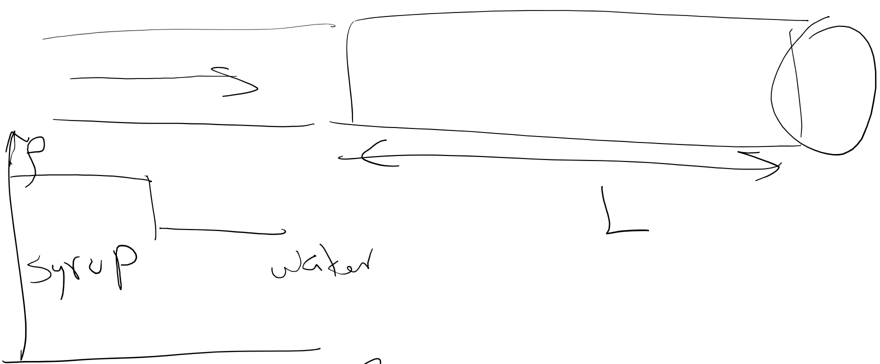
$$\frac{DJ}{Dt} = (\mathrm{div}\, v) J$$

$$\frac{DJ}{Dt} = (\operatorname{div} v)\, J = \underbrace{\left(\frac{\partial v_1}{\partial y_1} + \frac{\partial v_2}{\partial y_2} + \frac{\partial v_3}{\partial y_3}\right)}_{\operatorname{div} \hat{v}} J$$

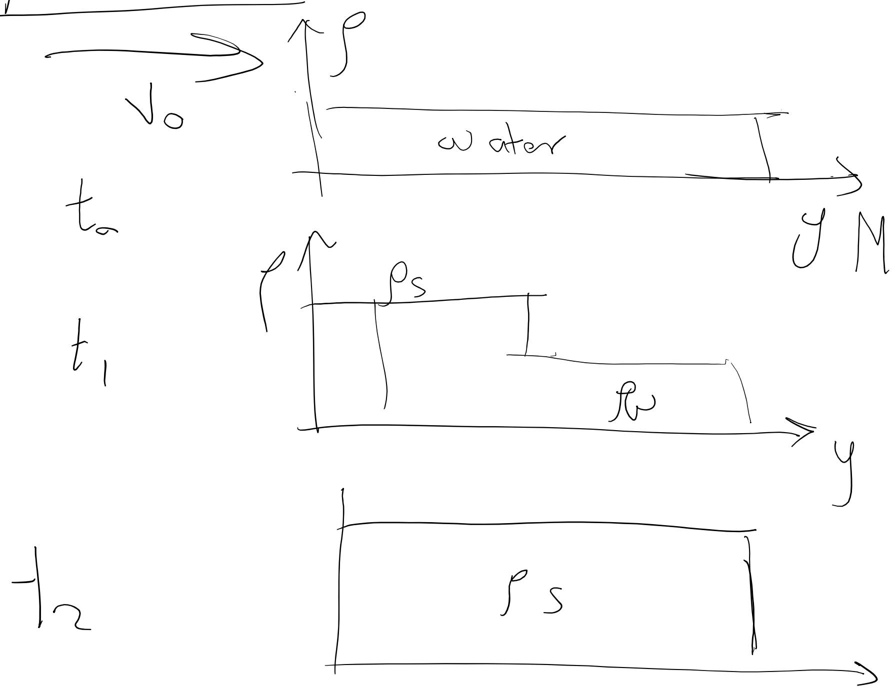
Balance laws: motivation

$F =$ rate (linear momentum)

Linear momentum $= \sum$ mass x velocity



area section $= A$



Linear momentum

$M_0 = \text{Mass} \times V_0 = (\rho_w A L) V_0$

$M_1 = \left(\left(\frac{\rho_w + \rho_s}{2}\right) A L\right) V_0$

$M_2 = \rho_s A L V_0$

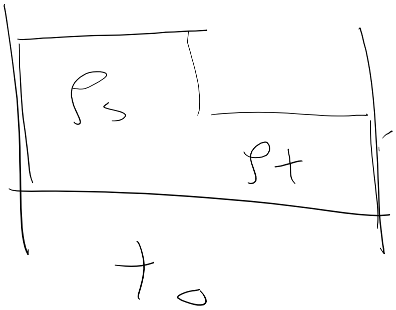
Change of linear momentum from t_0 to $t_2 > 0$

Its rate > 0 for t_0 to t_2

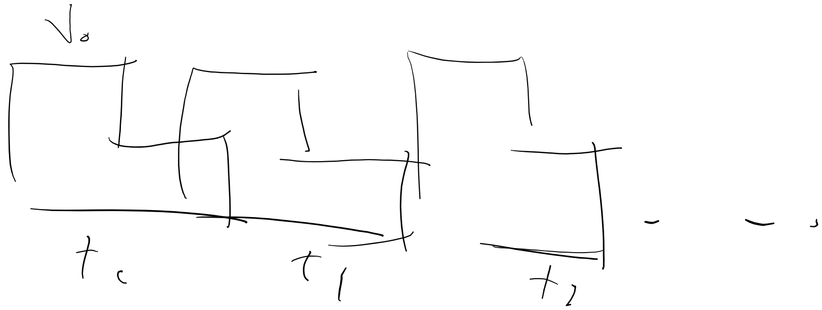
$\Rightarrow$ $F =$ rate of linear momentum > 0

Force must be zero (because we deal with constant velocity).

The problem is that Newton's law is for a fixed material blub. We basically follow material and need to take Lagrangian time derivate



follow the material $\longrightarrow$



always remains

Linear momentum $P = \left(\frac{\rho_s + \rho_w}{2}\right) A L v_0$

Constant $\Rightarrow \frac{DP}{Dt} = 0 = F$

whereas $\frac{\partial P}{\partial t} \neq 0 \neq F$

From Abeyaratne:

We say that Ω is an *extensive physical property* of the body if there is a function $\Omega(\cdot, t; \chi)$ defined on the set of all parts $\mathcal{P}$ of $\mathcal{B}$ which is such that

(i)

$$\Omega(\mathcal{P}_1 \cup \mathcal{P}_2, t; \chi) \overset{?}{=} \Omega(\mathcal{P}_1, t; \chi) + \Omega(\mathcal{P}_2, t; \chi) \tag{1.30}$$

for all arbitrary disjoint parts $\mathcal{P}_1$ and $\mathcal{P}_2$ (which simply states that the value of the property Ω associated with two disjoint parts is the sum of the individual values for each of those parts), and

(ii)

$$\Omega(\mathcal{P}, t; \chi) \to 0 \quad \text{as the volume of } \chi(\mathcal{P}, t) \to 0. \tag{1.31}$$

Under these circumstance there exists a *density* $\omega(p, t; \chi)$ such that

Volumetric value integral "Capital letter"

$$\boxed{\Omega(\mathcal{P}, t; \chi) = \int_{\mathcal{P}} \omega(p, t; \chi)\, dp.} \tag{1.32}$$

lower case density

Thus, we have the property $\Omega(\mathcal{P}, t; \chi)$ associated with parts $\mathcal{P}$ of the body and its density $\omega(p, t; \chi)$ associated with particles p of the body, e.g. the energy of $\mathcal{P}$ and the energy density at p.

Examples

Mass: $M = \int \rho \, dV$

mass density

Linear Momentum $P = \int p \, dV$

$p = \rho v$

linear momentum density

Energy $E = \int e_v \, dV$

Volumetric energy density

Balance laws:

$$\frac{D}{Dt}\int_{P_t}\rho \, dV$$

$$\frac{D}{Dt} M = 0$$

$$\frac{D}{Dt} P = F \quad \text{(sum of forces)}$$

$$\frac{dP}{dt} \neq F$$

For balance laws we deal with

$$\frac{D}{Dt}\Omega = \underbrace{\frac{D}{Dt}\int_{P_y}\omega(y,t)\,dV_y}_{\text{need to learn how to calculate}}$$

Example of integrals with moving boundary

$$\frac{D}{Dt}\int_{y_A(t)}^{y_B(t)} g(y,t)\,dy$$

$y_A(t_0)$ $y_B(t_0)$

$$G=\int_{y_A(t)}^{y_B(t)} g(y,t)\,dy$$

$G(t_1)$: $g(y,t_1)$ $y_A(t_1)$ $y_B(t_1)$ $\dot{y}_B\Delta t$

$G(t_2)$ $t_2=t_1+\Delta t$

$\dot{y}_A\Delta t$ $g(y,t_1+\Delta t)\simeq g(y,t_1)+\frac{\partial g}{\partial t}\Delta t$

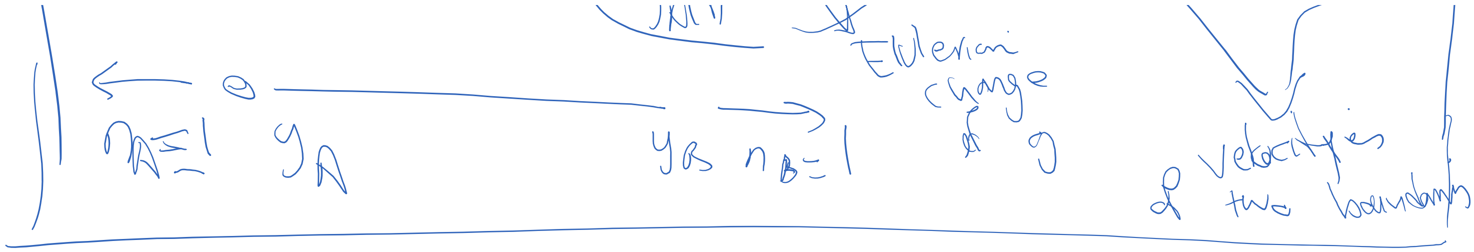
$y_A(t_1+\Delta t)\simeq y_A(t_1)+\dot{y}_A\Delta t$ $y_B(t_1+\Delta t)$ $y_B(t)+\dot{y}_B\Delta t$

$$\lim_{\Delta t\to 0}\frac{G(t_2)-G(t_1)}{\Delta t}=\int_{y_A}^{y_B}\frac{\partial g}{\partial t}dy-\dot{y}_A\, g(y_A)+\dot{y}_B\, g(y_B)$$

$$\boxed{\frac{D}{Dt}\int_{y_A(t)}^{y_B(t)} g(y,t)\,dt=\int_{y_A(t)}^{y_B(t)}\frac{\partial g}{\partial t}dy-\dot{y}_A\, g(y_A)+\dot{y}_B\, g(y_B)}$$

$$=\int_{y_A(t)}^{y_B(t)}\frac{\partial g}{\partial t}dy+n_A\cdot V_A g_A+n_B V_B\, g(y_B)$$

Eulerian change



$$\frac{D}{Dt}\int_{P_t} g(y,t)\,dV = \int_{P_t} \frac{\partial g}{\partial t}\,dV_y + \int_{\partial P_t} g\,\vec{v}\cdot n\,dS$$