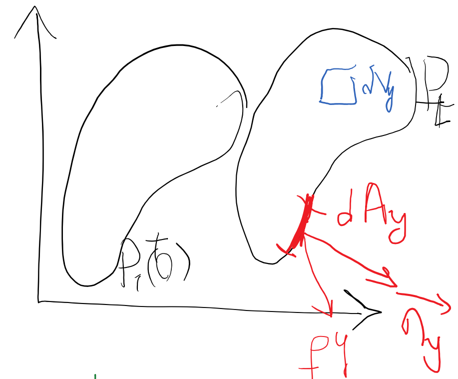
2019/10/28

Monday, October 28, 2019 11:42 AM

extensive property ← $S(t) = \int_{P_t} s \, dv$

s → some density



$$\frac{D}{Dt} S = \int_{P_t} \underbrace{r_s(y,t)}_{\text{Volumetric source term}} dV_y - \int f_s^y (y,t) \cdot \underbrace{n_y dA_y}_{d\vec{A}_y}$$

− outward

f_s^y = spatial flux density

= OUTWARD

= Always 1 tensor order higher than (S, s)

Tensor order	Balance law	S	s (Temporal flux)	Source term	Spatial flux
0	Mass	Mass	ρ (mass density)	NONE	NONE
1	Linear momentum	P	$p = \rho V$ (lin. mom. density)	ρb (body force)	− Stress $(-\sigma)$
0	Energy	E	$e_v = \rho e$ (energy density)	$\rho b \cdot v + Q +$	$q \quad -V \cdot \sigma$

$$\frac{DS}{Dt} = ?$$ find equation for this.

2.6 The Transport and Localization Theorems

Theorem 145 (Transport Theorem) *Let $g \in C^1(\Im, \Re)$ be a spatial scalar field. Then*

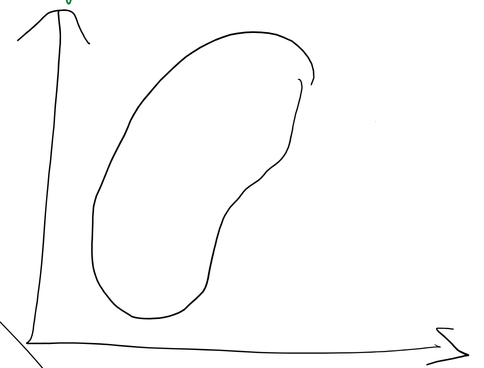
$$\underbrace{\frac{d}{dt}\int_{\mathcal{P}_t} g(\mathbf{y},t)\,dV_y}_{G} = \int_{\mathcal{P}_t}\left[\frac{\partial g}{\partial t}(\mathbf{y},t) + g_{,i}(\mathbf{y},t)\hat{v}_i(\mathbf{y},t) + g(\mathbf{y},t)\hat{v}_{i,i}(\mathbf{y},t)\right] dV_y \quad (1) \text{ Lagrangian } \frac{d}{dt} \to \frac{D}{Dt}$$

$$= \int_{\mathcal{P}_t}\left\{\frac{\partial g}{\partial t}(\mathbf{y},t) + [g\hat{v}_i]_{,i}(\mathbf{y},t)\right\} dV_y \quad (2)$$

$$= \underbrace{\int_{\mathcal{P}_t}\frac{\partial g}{\partial t}(\mathbf{y},t)\,dV_y}_{\text{change inside } \mathcal{P}_t} + \underbrace{\int_{\partial\mathcal{P}_t} g(\mathbf{y},t)\,[\hat{\mathbf{v}}(\mathbf{y},t)\cdot\mathbf{n}(\mathbf{y},t)]\,dA_y}_{\text{effect of moving boundary}}, \quad (3) \text{ Eulerian } \frac{\partial}{\partial t}$$

where $\mathbf{n}(\mathbf{y},t)$ is the outward unit normal to $\partial\mathcal{P}_t$ at $\mathbf{y}$.[20]

$$\frac{D}{Dt}\int_{P_t} g(y,t)\,dV_y = \int \frac{\partial g(y,t)}{\partial t}\,dV_y \;(\star) + \int_{\partial P_t} g\,\hat{v}\cdot n\,dA_y \;(\star\star)$$

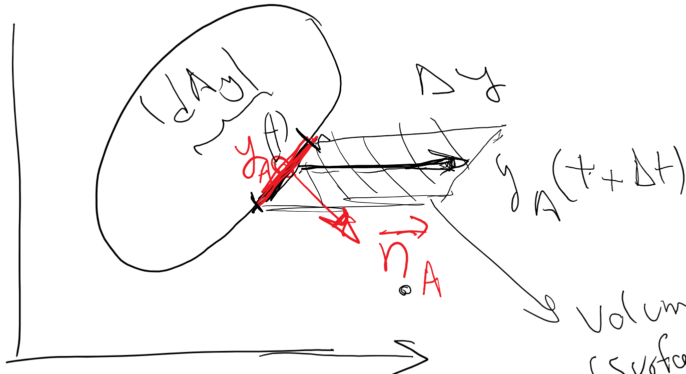


where is it coming from?

1D

$$\frac{D}{Dt}\int_{y_1(t)}^{y_2(t)} g(y,t)\,dy = \int_{y_1}^{y_2}\frac{\partial g}{\partial t}\,dy \;(\star) + \sum_{j=1}^{2} n_j V_j g_j \;(\star\star)$$

$n_1 \leftarrow y_1$ ———— $y_2 \; \xrightarrow{n_2}$ $\quad V_j = \dot{y}_j$



$$\text{volume (surface)} = \Delta y \cdot dA_y = \Delta y \cdot n\,dA_y$$

$$\Delta y = y_A(t+\Delta t) - y_A = \underbrace{\frac{dy_A}{dt}}_{v_A}\Delta t = v_A\Delta t$$

$$\text{Change of volume} = \left(\frac{\Delta y}{\Delta t}\cdot n\,dA_y\right)\Delta t$$

Change of volume $= \left(\frac{\Delta y}{\Delta t} \cdot n\, dA_y\right) \Delta t$

$\Delta t \to 0$ $\quad (V \cdot n\, dA_y)\, \Delta t$

we are adding $\{(v \cdot n\, dA_y)\, \Delta t\}\, g_A$ to G

Contribution from moving boundary

$$\frac{D}{Dt} G = \frac{D}{Dt}\int g\, dV_y = \int_{\partial P_t} g v \cdot n\, dA_y \longrightarrow \text{boundary contribution.}$$

we divide change by Δt

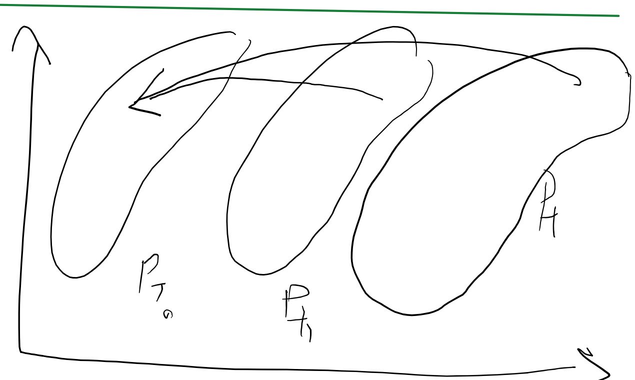
$\Delta t \to 0$

$$+ \int_{P_t} \frac{\partial g}{\partial t}\, dV_y$$

Formal Proof

$$G = \int_{P_t} g\, dV_y =$$

$P_t \longrightarrow$ variable



$$= \int_{P_0} g \underbrace{J\, dV_x}$$

move integral to reference configuration

P_{t_0} or $\longleftarrow P_0$

domain at $t = t_0$

$$J = \frac{dV_y}{dV_x} = \det F$$

$$\frac{DG}{Dt} = \frac{D}{Dt}\int g\, J\, dV_x$$

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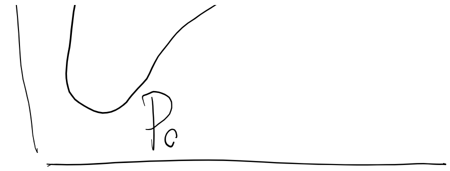
$P_0 \longrightarrow$ fixed $\Longrightarrow$



$$= \int \frac{D}{Dt}(gJ)\, dV_x \qquad (P_0 \text{ is fixed})$$

$$= \int \left(\frac{Dg}{Dt} J + g \frac{DJ}{Dt}\right) dV_x \longrightarrow \frac{DJ}{Dt} = \operatorname{div} \hat{v} \; J$$

$$\frac{DG}{Dt} = \int \left(\frac{Dg}{Dt} + g \operatorname{div} \hat{v}\right) \underbrace{J \, dV_x}_{dV_y}$$

$$\boxed{\frac{DG}{Dt} = \int_{P_t} \left(\frac{Dg}{Dt} + g \operatorname{div} \hat{v}\right) dV_y} \qquad \underline{L0}$$

$$\frac{Dg}{Dt} = \frac{\partial g}{\partial t} + \nabla_y g \cdot \hat{v} \qquad \frac{D}{Dt} \longleftrightarrow \frac{\partial}{\partial t}$$

$$\boxed{\frac{DG}{Dt} = \int_{D_t} \left(\frac{\partial g}{\partial t} + \underbrace{\nabla_y g}_{\text{grad}} \cdot \hat{v} + g \operatorname{div} \hat{v}\right) dV_y} \qquad \text{Line 1}$$

$$\frac{\partial g}{\partial y_i} \hat{v}_i + g \frac{\partial \hat{v}_i}{\partial y_i} = \frac{\partial\, g v_i}{\partial y_i} = \operatorname{div}(g\hat{v})$$

$$\boxed{\frac{DG}{Dt} = \int_{D_t} \left[\frac{\partial g}{\partial t} + \operatorname{div}(g\hat{v})\right] dV_y} \qquad L2$$

$$\boxed{\frac{DG}{Dt} = \int_{P_t} \Big[\frac{\partial g}{\partial t} + \underbrace{\mathrm{div}(g v)}\Big]\, dV_y} \quad (L2)$$

Gauss theorem

$$\boxed{\frac{DG}{Dt} = \int_{P_t} \frac{\partial g}{\partial t}\, dV + \int_{\partial P_t} g\, \hat{v} \cdot n\, dA_y} \quad (L3)$$

For L2 to L3 I have used Gauss theorem

$$\int_B f_{,i}\, dV \qquad \int_{\partial B} f\, n_i\, dA$$

$$\int_B \mathrm{div}\, f\, dV = \int_{\partial B} f\, n \cdot dA$$

Use this for $i = 1, 2, 3$

$y \to s$

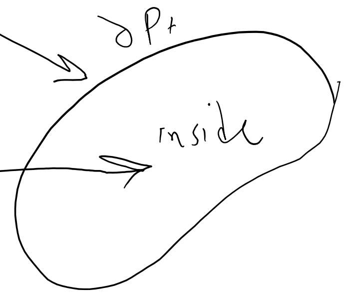
$$S = \int_{P_t} s\, dV_y$$

$$\frac{DS}{Dt} = \int_{P_t} r_s\, dV_y - \int_{\partial P_t} f_s^y\, n\, dA_y$$

$$\frac{DS}{Dt} = \int_{P_t} \frac{\partial s}{\partial t}\, dV_y + \int_{\partial P_t} \underbrace{s\, v \cdot n}\, dA_y \rightarrow (s \otimes v)\, \vec{n}$$

$$\frac{D}{Dt} \;=\; \int_{P_t} \frac{\partial}{\partial t}\, dV_y \;+\; \int_{\partial P_t} s\, v \cdot n\, dA_y$$

$$\Longrightarrow \int_{P_t} \left(\frac{\partial s}{\partial t} - r_s \right) dV_y + \int_{\partial P_t} \left(s \otimes v + f_s^y \right) \cdot n \, dA_y = 0$$

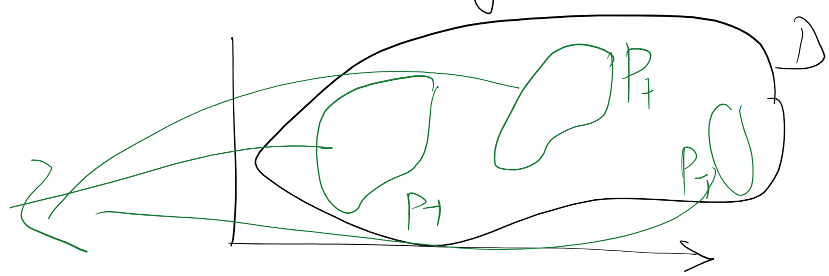


Use divergence theorem:

$$\int_{P_t} \left(\frac{\partial s}{\partial t} - r_s \right) dV_y + \int_{P_t} \operatorname{div}_o \left(s \otimes v + f_s^y \right) dV_y = 0$$

$$\Longrightarrow \forall P_t \quad \int_{P_t} \left[\frac{\partial s}{\partial t} - r_s + \operatorname{div}_o \left(s \otimes v + f_s^y \right) \right] dV_y = 0$$

sample P_t's

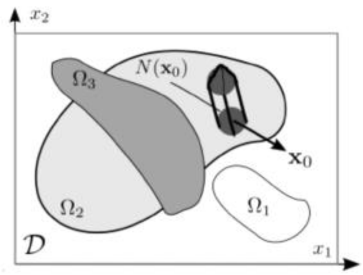


Localization theorem

Localization theorem

Localization theorem states that if the integral of a continuus function is zero for all subsets of $\mathcal{D}$, then the function is zero:

$$\boxed{\forall \Omega \subset \mathcal{D}: \int_{\Omega} \mathrm{g}(\mathbf{x})\, \mathrm{dv} = 0 \quad \Rightarrow \quad \forall x \in \mathcal{D}: \ g(\mathbf{x}) = 0} \tag{21}$$



Let's assume $g(x_0) \neq 0$ (e.g., $g(x_0) > 0$). Since $g(\mathbf{x})$ is continuus, there is a neighborhood of $\mathbf{x}_0$ ($N(\mathbf{x}_0)$) that $g(x) > 0$. We choose an Ω that is only nonzero inside $N(\mathbf{x}_0)$. Then, $\int_{\Omega} g(\mathbf{x})\, \mathrm{d}V > 0$. Thus, $g(\mathbf{x}_0)$ cannot be nonzero and the function g is identically zero.

PDE

Strong form of balance law

$$\underbrace{\frac{\partial s}{\partial t}}_{\text{temporal flux density}} + \underbrace{\nabla_y \cdot F_s^y}_{\text{spatial flux density}} = \underbrace{r_s^y}_{\text{source term}}$$

$$F_s^y = \underbrace{s \otimes \hat{v}}_{\text{Advective flux, convective}} + \underbrace{f^y}_{\text{non-convective flux or "diffusive flux"}}$$

Some comments about balance laws:

A)
Balance laws are general, whereas the PDE that we derive is not. The PDE only holds in the regions where the solution is smooth enough so

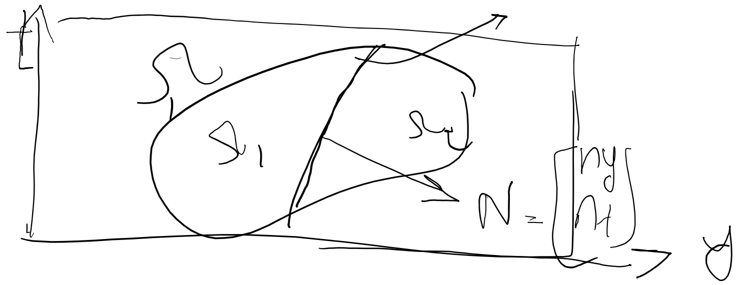
$$\int F_s^y n\, dA_y = \int \nabla_y \cdot f^y\, dV_y$$

A)
Balance laws are general, whereas the PDE that we derive is not. The PDE only holds in the regions where the solution is smooth enough so that we can apply the divergence theorem

$$\int_{\partial P_t} F^y_{n_0}\, dA_y = \int_{P_t} \nabla_y \cdot F^y \, dV_y$$

not always possible

if $\nabla_y \cdot F^y$ cannot be calculated & is not smooth



we can write balance law for $\Omega = \Omega_1 \cup \Omega_2$
Ω_1
Ω_2

We can derive the "jump conditions"

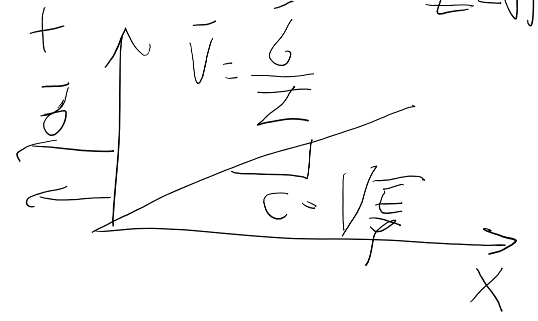
$$[F_{yt}] \cdot N_{yt} = 0 \implies \begin{bmatrix}[F_y] \\ [S]\end{bmatrix} \cdot \begin{bmatrix} n_y \\ n_t \end{bmatrix} = 0$$

$$\boxed{[F_y] \cdot n_y + [S] \cdot n_t = 0} \;\star$$

HW

Also called
Rankine Hugoniot condition in fluids

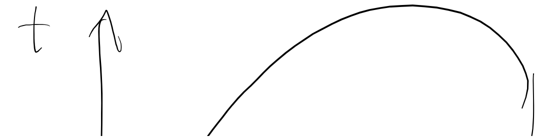
$Z = \sqrt{\rho E}$



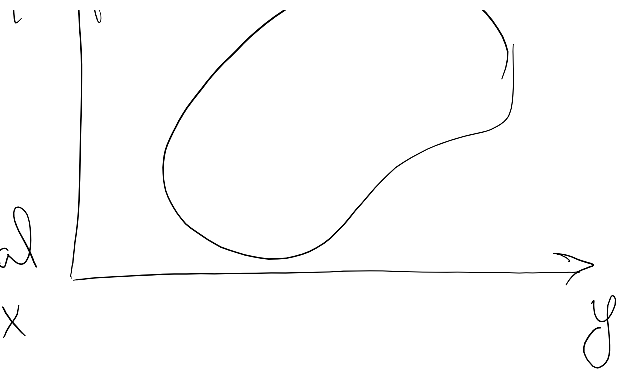
B) Balance laws in spacetime

$$\frac{\partial S}{\partial t} + \nabla_y \cdot F^y = r^S \qquad F^y = f^y + S(x)v$$

$\rightarrow [F^y]$ $\quad \nabla F^y$



$$\nabla_{st} \cdot \begin{bmatrix} F^y \\ s \end{bmatrix} = \nabla_y \cdot F^y + \frac{\partial s}{\partial t}$$

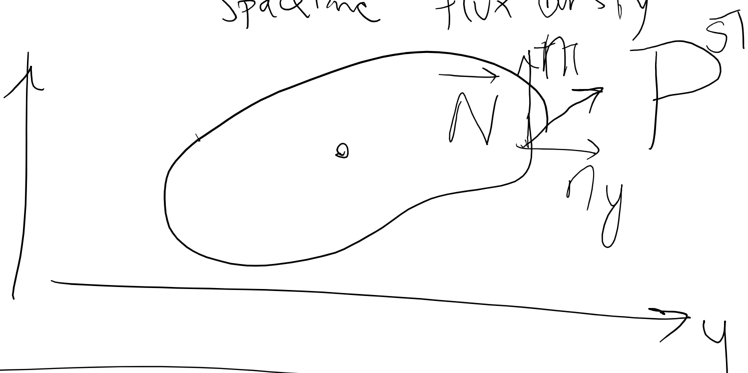


$$\Big(\underset{\text{space}}{\nabla_y} \quad \underset{\text{time}}{\frac{\partial}{\partial t}}\Big) \cdot \begin{bmatrix} F^y \\ s \end{bmatrix} \begin{matrix} \to \text{spatial flux} \\ \to \text{temporal flux} \end{matrix}$$

$$\frac{\partial s}{\partial t} + \nabla_y \cdot F^y = \underbrace{\nabla_{st}}_{\text{spacetime divergence}} \cdot \begin{bmatrix} F^y \\ s \end{bmatrix} = r^s$$

$$\boxed{\nabla_{st} \cdot F = r^s} \qquad \underset{\text{spacetime flux density}}{F} = \begin{bmatrix} F^y \\ s \end{bmatrix}$$

$$\int_{P^{st}} (\nabla_{st} \cdot F - r^s)\, dV = 0$$

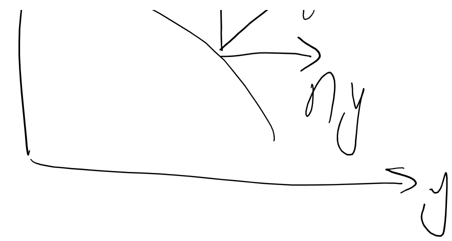


$$\boxed{\int_{\partial P^{st}} F \cdot N \, dA = \int_{P^{st}} r^s \, dV}$$

Expression of balance law in space time

$$F = \begin{bmatrix} F^y \\ s \end{bmatrix} \begin{matrix} \to \text{spatial flux} \\ \to \text{temporal flux} \end{matrix}$$

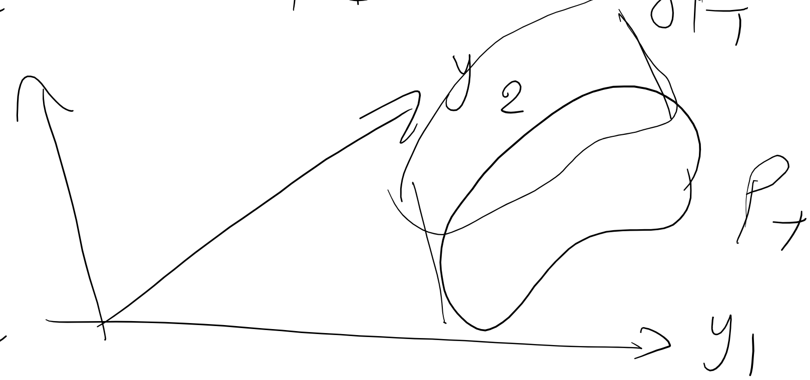
$$F = \begin{bmatrix} \vdots \\ s \end{bmatrix} \longrightarrow \text{temporal flux}$$



In comparison

$$\frac{DS}{Dt} = \frac{D}{Dt}\int_{P_t} s\, dv = \int_{P_t} r\, dv_y - \int_{\partial P_t} F^y n\, dA_y$$

Similar to this extruded space×time P_t geometry



we can write balance laws for P^{st}