2019/10/30

Wednesday, October 30, 2019 11:36 AM

B) General expression of balance laws in spacetime

$$\frac{D}{Dt}\int_{P_t} s\, dV_y = \int_{P_t} r_s\, dV_y - \int_{\partial P_t} f_s^y\, n\, dA_y$$

$$\Longrightarrow \underbrace{\int_{P_t} \frac{\partial s}{\partial t}}_{} + \int_{\partial P_t} F_s^y\, n\, dA_y = \int_{P_t} r_s\, dV_y$$

express it as temporal flux on time inflow & outflow for arbitrary shapes in space time

$$\boxed{\int_{\partial P} F\cdot N\, dA_y = \int_{P_t} r_s\, dV_y}$$

$$F = \begin{bmatrix} f_s^y \\ s \end{bmatrix}$$

$f_s^y \rightarrow$ space flux density ($F_s^y = \underbrace{s \otimes v}_{\text{advective part}} + \underbrace{f_s^y}_{\text{diffusive part}}$)

$s \rightarrow$ temporal flux = density of balanced quantity $\int$

$$N = \begin{bmatrix} \vec{n_y} \\ \hline n_t \end{bmatrix} \quad \underset{\text{in 2D}}{=} \quad \begin{bmatrix} n_{y_1} \\ n_{y_2} \\ n_t \end{bmatrix}$$

$\vec{n_y} \rightarrow$ vector

$n_t \rightarrow$ scalar

$$\int_{\partial P_t} F\cdot N\, dA_y = \int r_s\, dV_y \qquad \text{Balance law}$$

$\Longrightarrow$ ① PDE $\boxed{\nabla_{yt}\cdot F - r_s = 0}$

$$\begin{bmatrix} \nabla_y & \frac{\partial}{\partial t} \end{bmatrix} \begin{bmatrix} f_s^y \\ s \end{bmatrix} - r_s = \frac{\partial s}{\partial t} + \nabla_y \cdot f_s^y - r_s = 0$$

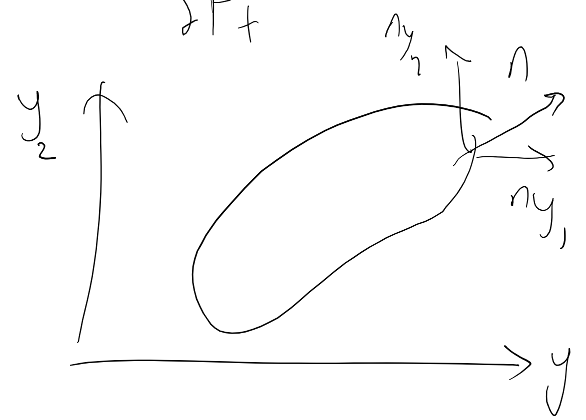
② $\boxed{[F]\cdot N = 0}$

$$[F_s^y]\cdot \vec{n}_y + [S]\cdot n_t = 0$$

Steady state or static balance laws, $\frac{D}{Dt}\ \frac{\partial}{\partial t} = 0$

no rate in time

$$\frac{D}{Dt}S = 0 = \int_{P_t} r_s\, dV - \int_{\partial P_t} f_s^y\, n\, dA_y$$

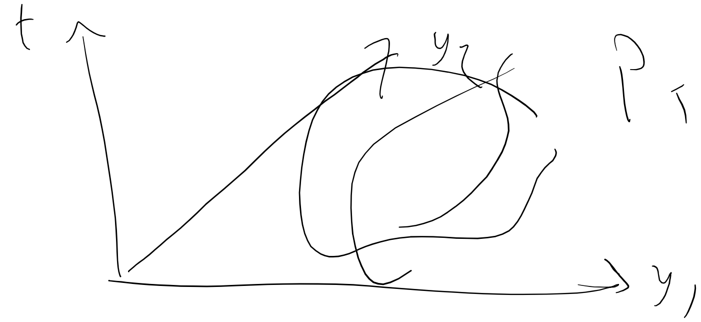


$$\oint_{\partial P_t} f_s^y\, n\, dA_y = \int_{P_t} r_s\, dV$$

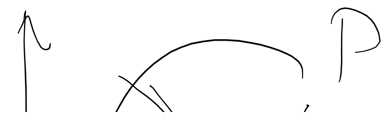
static flux

source

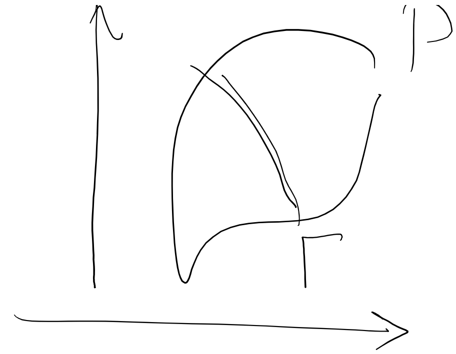
$$\oint_{\partial P_t} F\cdot N\, dA_y = \int_{P_t} r_s\, dV$$



For any balance law we have



$$\oint_{\partial P} F n \, dS = \int_P r \, dV$$



$$\Rightarrow \begin{cases} \nabla \cdot F - r = 0 & \text{PDE} \\ [F] \cdot n = 0 & \text{Jump condition} \end{cases}$$

Dynamic case

$F = \begin{bmatrix} F_s^y \\ S \end{bmatrix}$

$\left(\nabla_y \cdot F_s^y + \frac{\partial S}{\partial t}\right) - r_s = 0$

$\nabla \cdot F$

$[F_s^y] \cdot n_y + [S] \cdot n_t = 0$

PDE

$\nabla \cdot F - r = 0$

Jump

$[F] \cdot N = 0$

Static

$F = F_s^y$

$\nabla_y \cdot F_s^y - r = 0$

$[F_s^y] \cdot n_y = 0$

C) Balance laws for arbitrary z coordinate

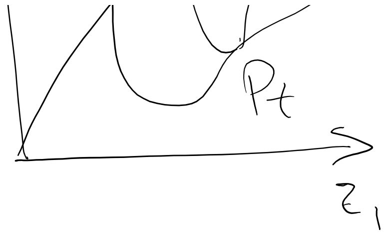
$z = x$

$z = y$

or something in between (Arbitrary Lagrangian Eulerian) ALE formulation

$\nabla_z (z_t) S = \int_{P_t} s \, dV_z$

DS ?



$$\frac{DS}{Dt}\Big|_{x\_fixed} = \ ?$$

we follow a fixed volume of material

$$\frac{DS}{Dt} = \int_{P_t} \frac{\partial s}{\partial t}\Big|_{z\text{-fixed}} dV_y + \int s\, \bar{V}(z,t)\, n_z\, dA_z$$

Eulerian $z=y \Rightarrow \bar{V} = \hat{v}$ —— $s\, \hat{v} \cdot n\, dA_y$

speed at which the boundary of P_t is moving when we follow a fixed material domain

Case 1 $z=y$ Eulerian

$$\bar{V}(z,t) = \frac{\partial y(x,t)}{\partial t}\Big|_{x\text{-fixed}} = V(x,t) = \hat{v}(y,t)$$

Case 2: $z=x$ Lagrangian

$$\bar{V}(z,t) = \frac{\partial x(x,t)}{\partial t}\Big|_{x\text{-fixed}} = 0$$

$z \neq x$ & $z \neq y$ $\quad \bar{V}(z,t) \neq 0$, $\neq \hat{v}(y,t)$

$$\frac{DS}{Dt}\Big|_{x\text{-fixed}} = \int_{P_t} r_s^z\, dV_z - \int f_s^z(z,t)\, n_z\, dA_z$$

$$\int_{P_t} \frac{\partial s}{\partial t}\Big|_{z\text{-fixed}} + \int_{\partial P_t} s\, \bar{V}(z,t)\, n_z\, dA_z = \int_{P_t} r_s^z\, dV_z - \int_{\partial P_t} f_s^z(z,t)\, n_z\, dA_z$$

P_t ... ∂P_t ... P_t ... ∂P_t

Interior terms

boundary terms

$$\int_{P_t}\left(\left.\frac{\partial s}{\partial t}\right|_{z\text{-fixed}} - r_s^z\right)dV_z + \int_{\partial P_t}\left(s\otimes V + f_s^z\right)\cdot n_z \, dA_z = 0$$

$$\checkmark \qquad + \int \nabla_z \cdot \left(s\otimes V + f_s^z\right) dV_z = 0$$

$$\forall P_t \quad \int_{P_t}\left[\frac{\partial s}{\partial t} - r_s^z + \nabla_z \cdot \left(s\otimes V + f_s^z\right)\right] dV_z = 0$$

arbitrary (Use localization to get)

$$\underbrace{\left.\frac{\partial s}{\partial t}\right|_{z\text{-fixed}}}_{\text{temporal flux density}} + \nabla_z \cdot \underbrace{F_s^z}_{\text{spatial flux density}} = \underbrace{r_s^z}_{\text{source term}}$$

$$F_s^z = \underbrace{s \otimes V_z}_{\text{velocity of boundary of } P_t} + f_s^z$$

" advective " "diffusive"

$V_z = \vec{v}$ $\quad z = y$

$V_z = 0$ $\quad z = x$

Conservation of mass:

$$\frac{DS}{Dt} = \frac{D}{Dt}\int_{P_t} s\, dV_y = \underbrace{\int_{P_t} r_s\, dV_y}_{\text{source term}} - \underbrace{\int_{\partial P_t} f_s^y \cdot n\, dA_y}_{\text{spatial flux}}$$

$$S = M \quad \frac{DM}{Dt} = \frac{D}{Dt}\int_{P_t} \underbrace{\rho}_{\text{mass density}}\, dV_y = 0 + 0$$

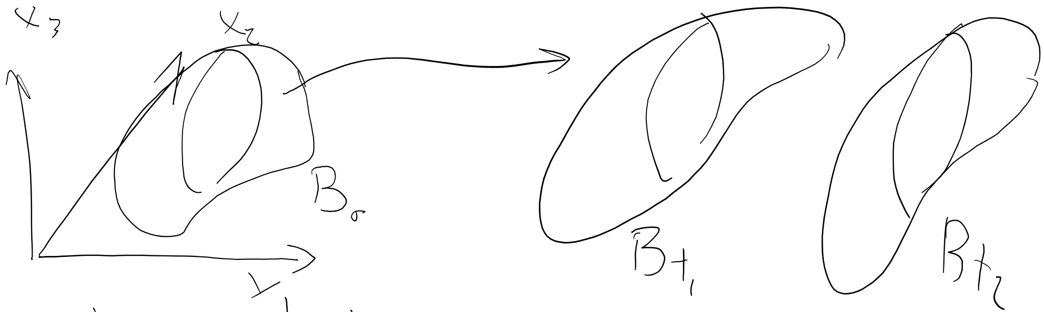
$$\frac{DM}{Dt} = 0 \qquad M = \text{const} \qquad \text{(conservation law)}$$

Conservation of Mass

Conservation of Mass

$$\frac{DM}{Dt} = 0 = \frac{D}{Dt}\int \rho \, dV_y$$

Lagrangian interpretation



$$M(0) = \int \underbrace{\rho_0(x)}_{\text{density @ } t=t_0} dV_x \qquad M(t) = \int_{B_t} \hat{\rho}(y,t)\, dV_y$$

$$\frac{dM}{dt} = 0 \longrightarrow M(t) = M(0)$$

$$\int_{B_t} \hat{\rho}(y,t)\, \underline{dV_y} = \int_{B_0} \rho_0(x)\, dV_x$$

$$dV_y = J\, dV_x$$

$$J = \det F$$

$$\int_{B_0} \rho(x,t)\, J\, dV_x = \int_{B_0} \rho_0(x)\, dV_x$$

$$\int_{B_0} \left(\rho(x,t) J - \rho_0(x)\right) dV_x = 0$$

B_0 arbitrary $\longrightarrow$ use localization

V arbitrary $\longrightarrow$ use localization

Balance of mass: Lagrangian Interpretation

$$\boxed{\rho J = \rho_0 \Longrightarrow \rho(x,t) = \frac{\rho_0(X)}{J(x,t)}}$$

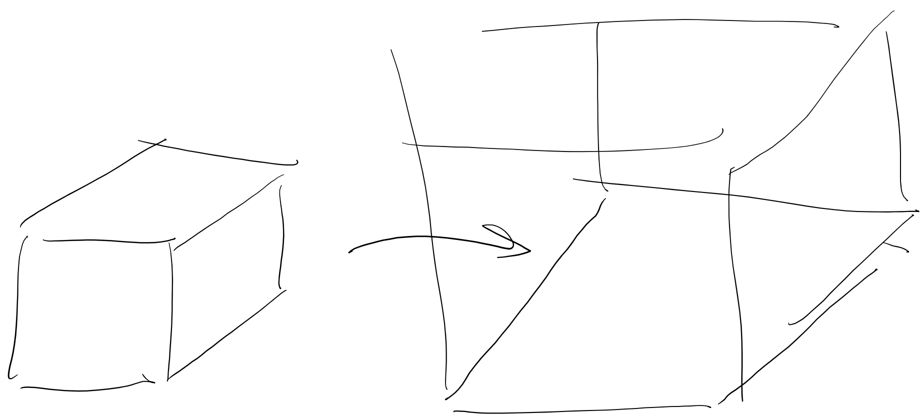
$$dm = \rho \, dV$$

at $t = t_0$ $\quad \rho_0 \, dV_X \quad \parallel \quad \Longrightarrow$

at t $\quad \hat{\rho}(y,t) \, dV_y = \rho(x,t) \, J \, dV_X$

$$\rho_0 = \rho J$$

$$\rho = \frac{\rho_0}{J}$$



dx $\qquad\qquad$ $1.1 \, dx$

$dV_x = dx^3$ $\qquad\qquad$ $dV_y = \underbrace{(1.1)^3}_{J} dx^3$

$dm = dV_x \rho_0$ $\qquad\qquad$ $dm = \rho \, dV_y$

$= \rho_0 \, dx^3 \quad = \quad = \rho \underbrace{(1.1)^3}_{J} dx^3$

$$\rho = \frac{\rho_0}{(1.1)^3} = \frac{\rho_0}{J}$$

Volume $\uparrow$ $\rho \searrow$

Uses of dm being constant in time

Theorem 151 (Reduced Transport Theorem) *Let* $g \in C^1(\Im, \Re)$. *Then*

$$\frac{d}{dt}\int_{\mathcal{P}_t} g(\mathbf{y},t)\rho(\mathbf{y},t)dV_y = \int_{\mathcal{P}_t}\left[\frac{\partial g}{\partial t}(\mathbf{y},t) + g_{,i}(\mathbf{y},t)\hat{v}_i(\mathbf{y},t)\right]\rho(\mathbf{y},t)dV_y.$$

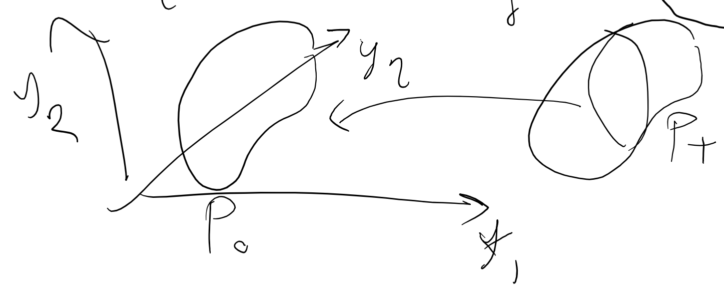
$$= \int_{\mathcal{P}_t}\left[\frac{\partial g}{\partial t}(\mathbf{y},t) + \nabla g(\mathbf{y},t)\cdot\hat{\mathbf{v}}(\mathbf{y},t)\right]\rho(\mathbf{y},t)dV_y$$

added to transport equation

Compare Transport

$$\frac{D}{Dt}\int g\,dV_y = \dots$$

$$\frac{D}{Dt}\int_{P_t}\hat{g}(y,t)\underbrace{\rho(y,t)\,dV_y}_{dm_y} = \frac{D}{Dt}\int_{P_0} g(x,t)\,dm =$$



$$= \int_{P_0}\frac{D}{Dt}\left(g(x,t)\,dm\right) = \int_{P_0}\frac{D}{Dt}g(x,t)\underbrace{dm}_{\text{doesn't depend on time}}$$

$$= \int_{P_t}\frac{D}{Dt}\hat{g}(y,t)\,dm$$

$$\left(\frac{\partial g}{\partial t} + \nabla_y g\cdot\hat{v}\right)\rho\,dV_y$$

$$= \int \underbrace{\left(\frac{\partial g}{\partial t} + \nabla_y g \cdot \hat{v}\right)}_{\frac{Dg}{Dt}} \rho^y \, dV_y$$

$$\begin{aligned} \frac{d}{dt}\int_{\mathcal{P}_t} g(\mathbf{y},t)\,dV_y &= \int_{\mathcal{P}_t}\left[\overbrace{\frac{\partial g}{\partial t}(\mathbf{y},t) + g_{,i}(\mathbf{y},t)\hat{v}_i(\mathbf{y},t)}^{\frac{Dg}{Dt}} + g(\mathbf{y},t)\hat{v}_{i,i}(\mathbf{y},t)\right] dV_y \\ &= \int_{\mathcal{P}_t}\left\{\frac{\partial g}{\partial t}(\mathbf{y},t) + \left[g\hat{v}_i\right]_{,i}(\mathbf{y},t)\right\} dV_y \\ &= \int_{\mathcal{P}_t}\frac{\partial g}{\partial t}(\mathbf{y},t)\,dV_y + \int_{\partial\mathcal{P}_t} g(\mathbf{y},t)\left[\hat{\mathbf{v}}(\mathbf{y},t)\cdot\mathbf{n}(\mathbf{y},t)\right] dA_y, \end{aligned}$$

Transport eqn

$\rightarrow$ we don't have this term that came from

$$\frac{D}{Dt} J = \mathrm{div}_y \vec{v}\, J$$

Can prove reduced transport eqn by letting $\tilde{g} = \rho g$ in transport eqn

$$\frac{D}{Dt}\int \underbrace{g}\, dV_y = \ldots \qquad \text{Transport eqn}$$

g is volumetric density

$$\frac{D}{Dt}\int g\rho \, dV_y = \frac{D}{Dt}\int g \, dm = \text{Reduced Transport eqn}$$

g is mass density of G

Conservation of Mass in Eulerian framework:

$$\frac{D}{Dt}\int_{P_t} \underbrace{\rho}\, dV_y = \int_{P_t} \frac{\partial \rho}{\partial t} dV_y + \int_{\partial P_t} \rho \hat{v}\cdot n \, dA_y$$

$$\frac{D}{Dt}\int s \, dV_y = \int \frac{\partial s}{\partial t} dV_y + \int s \hat{v}\cdot n \, dA_y$$

$$\frac{D}{Dt}\int_{P_t} s\,dV_y = \int_{P_t} \frac{\partial s}{\partial t}\,dV_y + \int_{\partial P_t} s\,\hat{v}\cdot n\,dA_y$$

$$\frac{D}{Dt}\int \rho\,dV_y = \frac{D}{Dt} M = \int_{P_t} \cancel{r\rho}\,dV_y + \int_{\partial P_t} \cancel{f_y^{\rho}}\,n\,dA_y$$

$$\boxed{\frac{DM}{Dt} \int_{P_t} \rho\,dV_y = \int_{P_t} \frac{\partial \rho}{\partial t} + \int_{\partial P_t} \rho\,\hat{v}\,n\,dA_y = 0}$$

apply divergence theorem

$$\int_{P_t} \frac{\partial \rho}{\partial t} + \int_{P_t} \underbrace{\mathrm{div}_y}_{\nabla_y \cdot}\,\rho\,\hat{v}\,dV_y = 0$$

$$\int_{P_t} \left[\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho\,\hat{v})\right] dV_y = 0$$

localization

Balance of mass in Eulerian framework

$$\frac{\partial \rho}{\partial t} + \underbrace{\mathrm{div}}_{\nabla_y \cdot}\underbrace{(\rho\,\hat{v})}_{\text{spatial flux}} = 0$$

Having a full divergence is good because:

1. We have a definition of spatial flux density (what div acts on)
2. Numerical formulation of many methods such as Discontinuous Galerkin method becomes much simpler

s

$$\frac{Ds}{Dt} = \frac{\partial s}{\partial t} + \nabla_y s \cdot \hat{v}$$

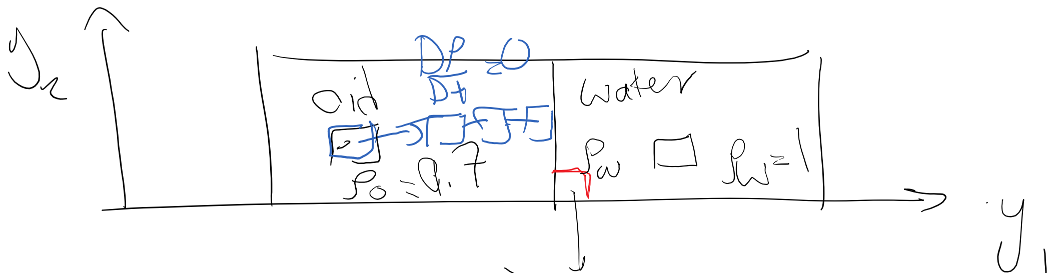
Another way to write conservation of mass

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \hat{v}) = \frac{\partial \rho}{\partial t} + \frac{\partial \rho \hat{v}_i}{\partial y_i} =$$

$$\underbrace{\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial y_i}\hat{v}_i} + \rho \frac{\partial \hat{v}_i}{\partial y_i} = \underbrace{\left(\frac{\partial \rho}{\partial t} + \nabla_y \rho \cdot \hat{v}\right)}_{\text{Material rate}} + \rho(\mathrm{div}\,\hat{v}) =$$

$$\boxed{\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \hat{v}) = \frac{D\rho}{Dt} + \underbrace{\rho(\mathrm{div}\,\hat{v})}_{\text{not a complete div term :(}} = 0}$$

For incompressible fluid $\frac{D\rho}{Dt} = 0$ $\quad(\rho = \text{const})$



$\frac{\partial \rho}{\partial t} < 0$ Eulerian

Incompressible fluid

$$\cancel{\frac{D\rho}{Dt}} + \rho\,\mathrm{div}\,v = 0 \qquad \boxed{\mathrm{div}\,\hat{v} = 0}$$

$$\frac{DJ}{Dt} = (\mathrm{div}\,\hat{v}) \cdot J$$

$$\frac{DJ}{Dt} =$$

$$\frac{D\rho}{Dt} = 0$$

Volume doesn't change $\quad dV_y = dV_x$

In solid mechanics we assume incompressibility for plastic deformation
(Poisson ratio = 0.5)