2019/11/04

Monday, November 4, 2019 11:38 AM

From last time:

Summary: conservation of mass:

Lagrangian: $\rho(x,t) = \dfrac{\rho_0(x)}{J(x,t)}$ if incompressible $J(x,t)=1 \Rightarrow \rho(x,t)=\rho_0(x)$

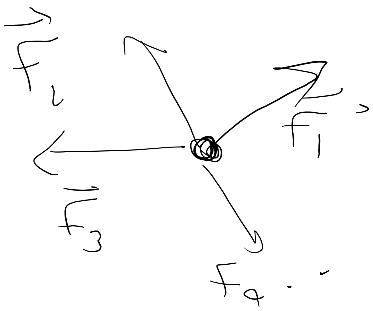
Eulerian: $\dfrac{\partial\hat{\rho}(y,t)}{\partial t} + \operatorname{div}(\rho\,\hat{v}(y,t)) = \dfrac{D\rho(y,t)}{Dt} + \rho \operatorname{div}\hat{v} = 0$

if incompressible $\dfrac{D\rho}{Dt} = 0 \Rightarrow \operatorname{div}\hat{v} = \dfrac{\partial v_1}{\partial y_1} + \dfrac{\partial v_2}{\partial y_2} + \dfrac{\partial v_3}{\partial y_3} = 0$

Balance of linear momentum:

$F = \sum F$

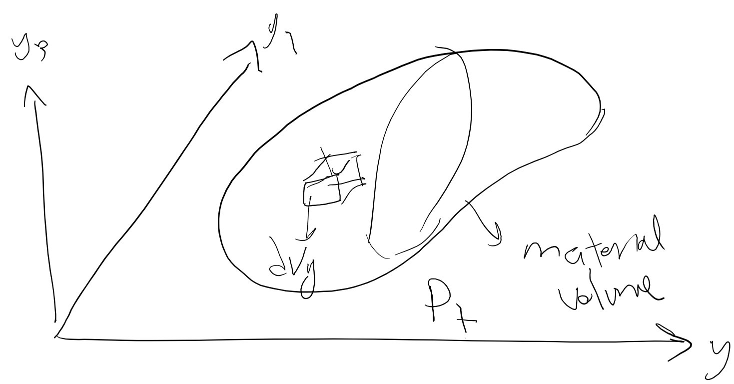
$= ma = \dfrac{DP}{Dt}$



P = linear momentum $= mv$ = mass x velocity

For a continuum

$\underbrace{\dfrac{DP}{Dt}}_{\text{linear momentum}} = \underbrace{\sum F}_{\text{sum of forces}}$



P = sum of mass x velocity

$= \int \vec{v}\, dm$ $\quad dm = \rho\, dV_y$

$\dfrac{D}{Dt}\left(\int_{P_t} \rho \vec{v}\, dV_y \right) \quad \int \rho\, dV_y$

$$\boxed{P = \int_{P_t} \underbrace{\rho \vec{v}}_{p} \, dV_y}$$

$$S = \int_{P_t} s \, dV_y$$

S → total quantity (mass, energy, linear momentum)

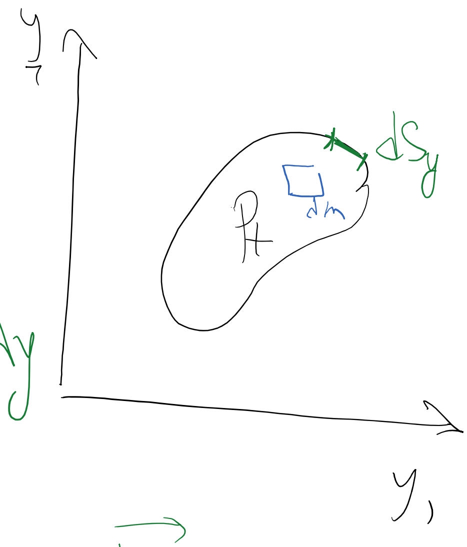
s → volumetric density of S

$\vec{p} = \rho \vec{v}$ = linear momentum density

($\vec{m}$ is also used in fluids)

$$\frac{DP}{Dt} = \frac{D}{Dt} \int_{P_t} p \, dV_y = F_{volumetric} + F_{surface}$$

$$= \int_{P_t} \vec{b} \, \underbrace{dm} + \int_{\partial P_t} \vec{t} \, dA_y$$



$\vec{b}$ = body force (force per unit mass) $dm = \rho \, dV_y$

e.g. from gravity $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix}$

unit of b is acceleration

traction $= \lim_{dA_y \to 0} \dfrac{d\vec{F}_{surface}}{dA_y}$

$$\boxed{\frac{DP}{Dt} = \frac{D}{Dt} \int_{P_t} p \, dV_y = \int_{P_t} \rho b \, dV_y + \int_{\partial P_t} \vec{t} \, dA_y}$$

we will see

$\vec{t} = \sigma \, \vec{n}$ → normal vector current

$$\vec{t} = \sigma\, \vec{n}_y$$

$\vec{t}$: vector

σ: Cauchy stress tensor

$\vec{n}_y$: normal vector in current configuration

T is used in coursenet

$$\frac{DP}{Dt} = \frac{D}{Dt}\int_{P_t} \rho\, dV_y = \int_{P_t} \rho b\, dV_y - \int_{\partial P_t} -\sigma \cdot n\, dA_y$$

$$\frac{DS}{Dt} = \frac{D}{Dt}\int_{P_t} s\, dV_y + \int_{P_t} r_s\, dV_y - \int_{\partial P_t} f_s^y \cdot n\, dA_y$$

r_s: source term

f_s^y: outward spatial flux corresponding to s

$S = P =$ linear momentum

$s = \vec{p} = \rho \vec{v}$ " " density

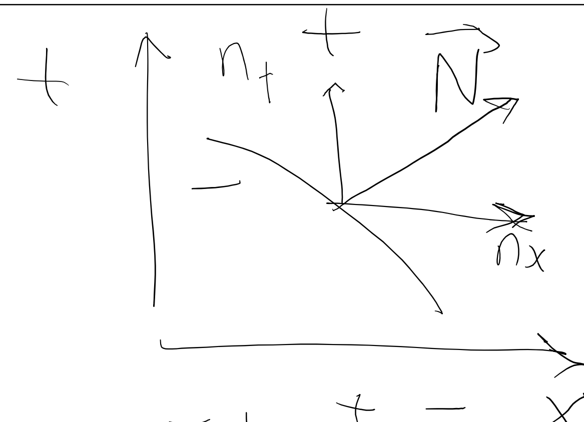
$r_p = \rho \vec{b}$ source term (body force)

$f_p^y = -\sigma$ outward spatial flux

HW

$$M = \begin{bmatrix} f_s^y \\ s \end{bmatrix} = \begin{bmatrix} -\sigma \\ p \end{bmatrix}$$

M: spacetime flux



$$[M] \cdot \vec{N} = 0$$

$$\boxed{[-\sigma] \cdot n_x + [p]\, n_t = 0}$$

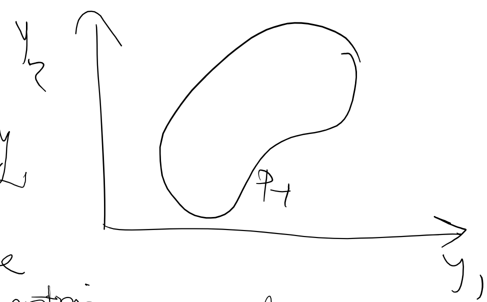
$$[\sigma] = \sigma^+ - \sigma^-$$

$$[p] = p^+ - p^-$$

Derivation of PDE for balance of linear momentum

Derivation of PDE for balance of linear momentum

$$\frac{D}{Dt}\int_{P_t} p\, dV_y = \int_{P_t} \frac{\partial p}{\partial t}\, dV_y + \int_{\partial P_t} p\, \underbrace{v \cdot n\, dA_y}_{\text{the rate of motion on the boundary of } P_t}$$



$$p\, v \cdot n = p_i e_i\, v_j n_j$$

Note $(p \otimes v)\vec{n} = \boxed{(p_i v_j e_i \otimes e_j)}\, n_k e_k$

$$= p_i v_j e_i \underbrace{(e_j \cdot e_k)}_{\delta_{jk}} n_k$$

$$= p_i v_j e_i n_j$$

$$\boxed{\frac{D}{Dt}\int_{P_t} p\, dV_y = \int_{P_t} \frac{\partial p}{\partial t}\bigg|_{y\text{-fixed}} dV_y + \underbrace{\int_{\partial P_t} \underbrace{p \otimes v}_{} \cdot n\, dA_y}_{}}$$

vector 2nd order tensor

$$D\int_{P_t} p\, dV_y = \int_{P_t} \rho b\, dV_y + \int_{\partial P_t} \sigma \cdot n\, dA_y$$

$$\int_{P_t}\left(\frac{\partial p}{\partial t} - \rho b\right) dV_y + \int_{\partial P_t} \underbrace{(p \otimes v - \sigma)}_{f_p^y} \cdot n\, dA_y = 0$$

outward flux of linear momentum $= \underbrace{p \otimes v}_{\text{advective part}} \; \underbrace{- \sigma}_{\text{diffusive part.}}$

Apply the divergence theorem

$$\forall P_t \quad \int_{P_t} \left\{ \frac{\partial p}{\partial t} - \rho b + \operatorname{div}(p \otimes v - \sigma) \right\} dV_y = 0$$

P_t is arbitrary, use localization

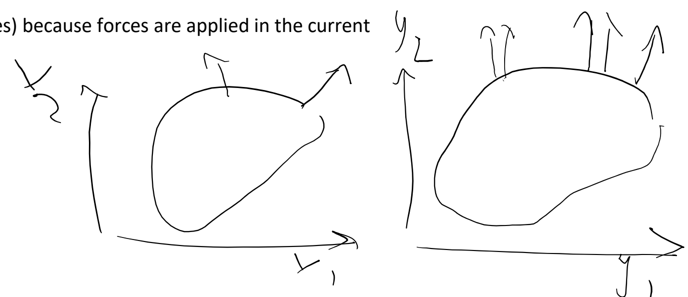
$$\boxed{\frac{\partial p}{\partial t} + \operatorname{div}(p \otimes v - \sigma) = \rho b} \qquad (*)$$

balance of linear momentum

$p = \rho v$

This equation is the balance of linear momentum in current configuration.

- We need to use the current configuration to express balance of linear momentum (forces) because forces are applied in the current configuration.
- However, we need to be careful in that for balance laws P_t (domain) follows a fixed Domain of material.

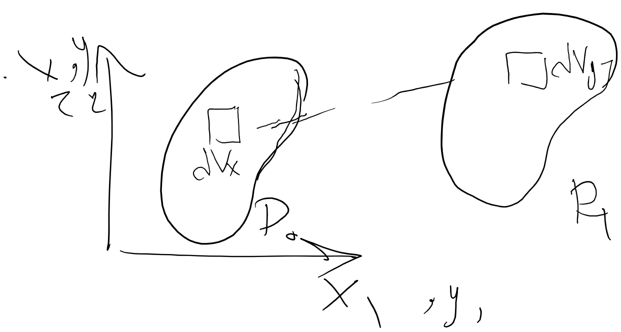


need to write balance of forces here

Equation (*) is perfectly fine for fluids since all equations are already written in current configuration. However for solids, we prefer to write all the equations in the referential configuration x.

$$\frac{D}{Dt} \int_{P_t} p \, dV_y = \int_{P_t} \rho b \, dV_y + \int_{\partial P_t} \sigma \cdot \vec{n}_y \, dA_y$$

(1) (2) (3)



terms ① & ② $\quad dV_y = J\,dV_x$

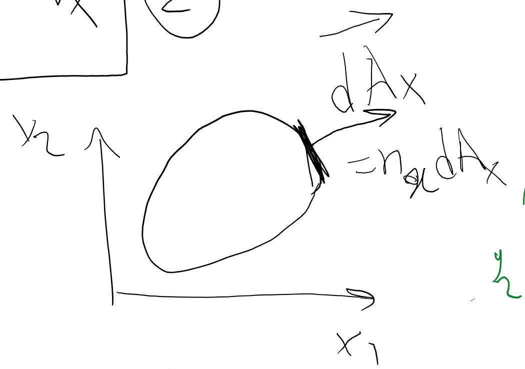
term ①: $\quad \frac{D}{Dt}\int_{P_0} \rho J\, dV_x = \frac{D}{Dt}\int_{P_0} (\rho J)\vec{v}\, dV_x = \frac{D}{Dt}\int_{P_0} \underbrace{\rho_0(x)\,\vec{v}}\, dV_x$

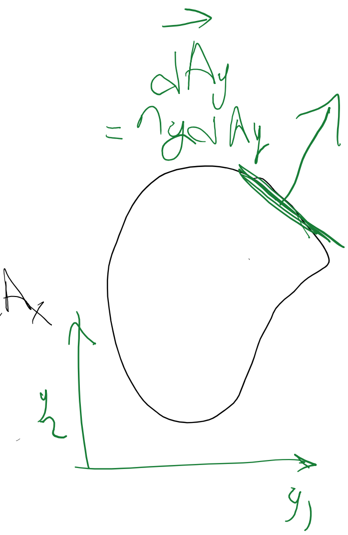
$$\underbrace{p(x,t)}_{\text{referential linear momentum density}} = \rho_0(x)\,\vec{v}(x,t)$$

$$\boxed{\frac{D}{Dt}P = \int_{P_0} \frac{D}{Dt}\, p(x,t)\, dV_x} \quad ①$$

② $\quad \int_{P_t} \rho\vec{b}\, dV_y = \int_{P_0} \rho\vec{b}\, J\, dV_x = \int \underbrace{(\rho J)}_{\rho_0(x)}\vec{b}\, dV_x$

$$\boxed{\int_{P_t} \rho b\, dV_y = \int \rho_0(x)\,\vec{b}(x,t)\, dV_x} \quad ②$$





③ $\quad \int_{\partial P_t} \sigma\cdot \underbrace{n_y\, dA_y}_{\vec{dA_y}}$

$$\vec{dA_y} = \underbrace{(J\, F^{-t})}_{\text{we showed this relation in kinematics section}}\vec{dA_x}$$

③ $\quad \int_{\partial P_t} \sigma\, \vec{dA_y} = \int_{\partial P_0} \sigma J F^{-t}\cdot \vec{dA_x} = \int (J\sigma F^{-t})\cdot n_x\, dA_x$

Piola-Kirchhoff stress tensor I (PK-I)

$$\int_{\partial P_t} \vec{\sigma} \, dA_y = \int_{\partial B_o} S_x n_o \, dA_x$$

(3)

$$S(x,t) = J(x,t)\,\sigma(y(x,t),t)\,F^{-t}(x,t)$$

PK-I tensor — 1st Piola-Kirchhoff stress tensor

$$\frac{D}{Dt}\int_{P_t} \rho \, dV_y = \int_{P_t} \rho b \, dV_y + \int_{\partial P_t} \sigma \cdot \vec{n}_y \, dA_y$$

(1) (2) (3)

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$\frac{D}{Dt}\int_{P_o} \rho_o \, dV_x = \int_{P_o} \rho_o b \, dV_x + \int_{\partial P_o} S \cdot \vec{n}_x \, dA_x$$

$$S = J\sigma F^{-t} \qquad \text{PK I}$$

→ Balance law in reference configuration

$$\int_{P_o} \frac{D}{Dt}\rho_o \, dV_x = \int_{P_o} \rho_o b \, dV_x + \int_{\partial P_o} S \cdot \vec{n}_x \, dA_x$$

(★★)

PDE for balance of linear momentum in Lagrangian configuration

(★★) Apply the divergence theorem

$$\int_{P_o} \frac{D}{Dt}\rho_o \, dV_x = \int \rho_o b \, dV_x + \int_{P_o} \underbrace{\mathrm{Div}_o}_{\substack{\text{referential} \\ \nabla_{x^o}}} S \, dV_x$$

$$\Rightarrow \forall P_o \quad \int \left(\frac{D}{Dt}\rho_o - \rho_o b - \mathrm{Div}_o S\right) dV_x = 0$$

$$\Rightarrow \forall \mathcal{P}_0 \quad \int_{\mathcal{P}_0} \left(\frac{D}{Dt} \vec{p}_0 - \rho_0 b - \mathrm{Div}\, S \right) dV_x = 0$$

$$\Rightarrow \boxed{\underbrace{\frac{D}{Dt} \vec{p}_0}_{\rho_0 \vec{a}} - \mathrm{Div}\, S = \rho_0 b}$$

Equation of Motion (EOM) in Lagrangian conf.

$$\vec{p}_0 = \rho_0(X)\, \vec{V}(X,t)$$

$$\frac{D}{Dt} \vec{p}_0 = \rho_0(X) \frac{D V(X,t)}{Dt} = \rho_0(X)\, \vec{a}(X,t)$$

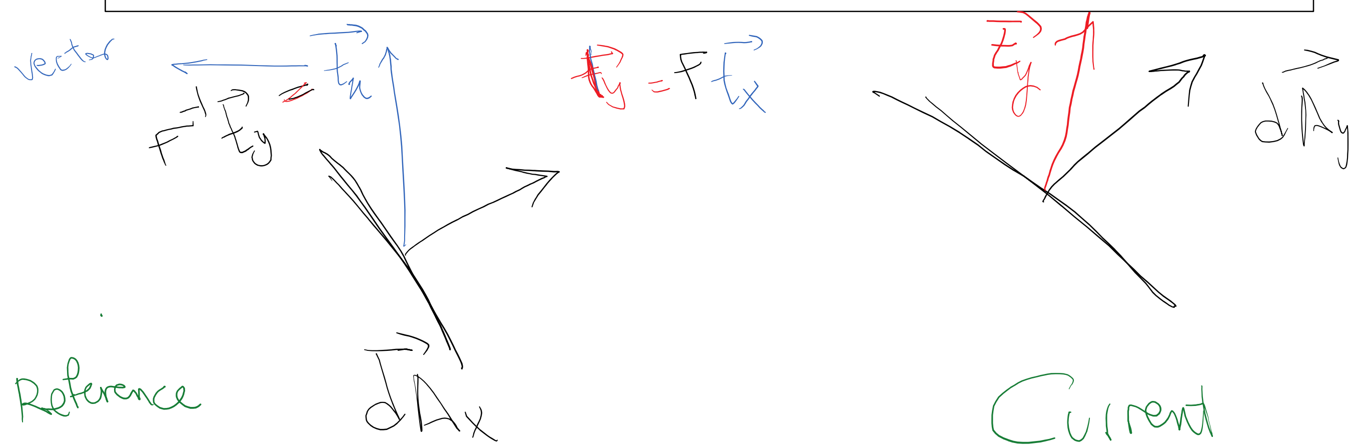
$$\mathrm{Div}\, S = S_{ij,j}\, e_i \qquad \begin{bmatrix} S_{11,1} + S_{12,2} + S_{13,3} \\ S_{21,1} + S_{22,2} + S_{23,3} \\ S_{31,1} + S_{32,2} + S_{33,3} \end{bmatrix}$$

$$\rho_0 \ddot{u}_1 - (S_{11,1} + S_{12,2} + S_{13,3}) = \rho_0 b_1$$

$$\rho_0 \ddot{u}_2 - (S_{21,1} + S_{22,2} + S_{23,3}) = \rho_0 b_2$$

$$\rho \ddot{u}_3 - (S_{31,1} + S_{32,2} + S_{33,3}) = \rho_0 b_3$$

e.g. $S_{11,1} = \dfrac{\partial S_{11}}{\partial X_1}$



	$\vec{t}_x$	$\vec{t}_y$
$d\vec{A}_x$	$\vec{t}_x = P\, d\vec{A}_x$	$\vec{t}_y = S\, d\vec{A}_x$
$d\vec{A}_y$		$\vec{t}_u = \sigma\, d\vec{A}$

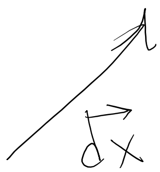
$S = JGF^{-t}$

PK-I

$\vec{dA_y}$		$\vec{t_y} = \sigma\, \vec{dA_y}$	PK-I

stress tensor takes surface differential & returns traction
(dA_x or dA_y) ($\vec{t_x}$ or $\vec{t_y}$)

P = PK-II stress tensor



$$\vec{dy} = F\, \vec{dx}$$

$\hat{}$

$\vec{dy}$

$\boxed{\text{vectors}}$ $dx = F^{-1} dy$

$$\vec{dA_y} = J F^{-T} dA_x$$

$\boxed{\text{covectors}}$ surfaces