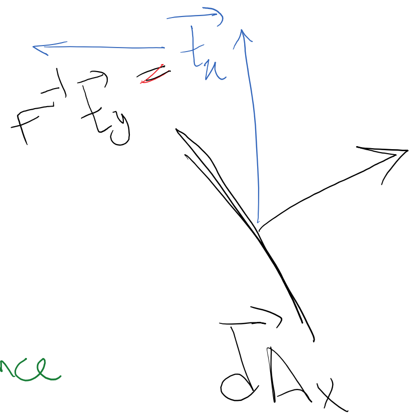
2019/11/06

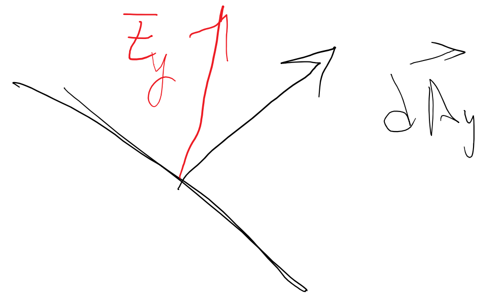
Wednesday, November 6, 2019 11:36 AM

vector

$F^{-1}\vec{t}_y = \vec{t}_x$

$\vec{t}_y = F\vec{t}_x$





Reference

Current

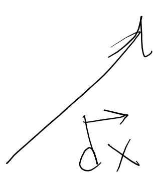
	$\vec{t}_x$	$\vec{t}_y$
$\vec{dA}_x$	$\vec{t}_x = P\,\vec{dA}_x$	$\vec{t}_y = S\,\vec{dA}_x$
$\vec{dA}_y$		$\vec{t}_y = \sigma\,\vec{dA}_y$

$S = JGF^{-t}$

PK-I

stress tensor takes surface differential (dAx or dAy) & returns traction ($\vec{t}_x$ or $\vec{t}_y$)

P = PK-II stress tensor



$\vec{dy} = F\,\vec{dx}$

$\hat{\vec{dy}}$

$\boxed{\text{vectors}}$ $dx = F^{-1}dy$

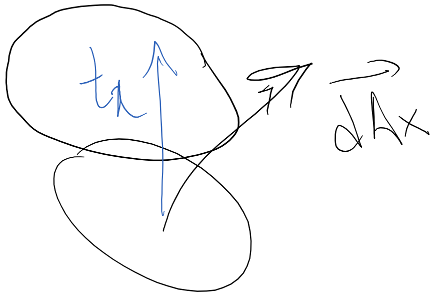
$\vec{dA}_y = JF^{-t}\,dA_x$

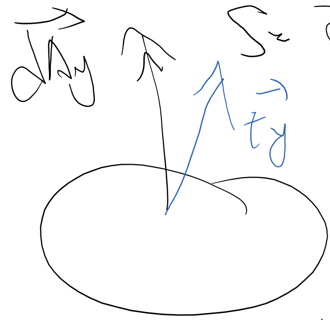
$\boxed{\text{covectors}}$ surfaces

$\boxed{\text{Covectors}}$ surfaces

$$s_y = \sigma\ \vec{dA_y} \longrightarrow s_y = \underbrace{S}_{PK\text{-}I}\ \vec{dA_x}$$

$$S = J\sigma F^{-t} \quad PK\text{-}I$$





vectors
$$t_y = F t_x$$
$$\vec{dy} = F\vec{dx}$$

$$\underbrace{dA_y = J F^{-t} dA_x}_{\text{covector}}$$

$$\vec{t_x} = F^{-1}\vec{t_y}$$

$$\vec{t_x} = F^{-1}\left(\underbrace{S}_{PK\text{-}I}\ dA_x\right) \qquad S = J\sigma F^{-t}$$

$$\vec{t_x} = F^{-1}(J\sigma F^{-t})\, dA_x \Longrightarrow$$

scalar

$$\vec{t_x} = \left(\underbrace{J F^{-1}\sigma F^{-t}}_{P}\right) \vec{dA_x}$$

P Piola-Kirchhoff II (PK-II) stress tensor

$$\boxed{P = J F^{-1}\sigma F^{-t}} \qquad P^t = J\,(F^{-t})^t\,\sigma^t\,(F^{-1})^t$$
$$= J\, F^{-1}\, \underbrace{\sigma}\, F^{-t}$$

we'll see σ is symmetric because of balance of linear momentum

		referential (x)	spatial (y)
Cauchy σ (T)	Area (normal)		n_y
	traction		s_y
PK-I S	Area	n_x	
	Traction		s_y
PK-II P	Area	n_x	
	Traction	s_x	

$s_y = \sigma\, n_y$

$s_y = S\, n_x \qquad S = J\sigma F^{-T}$

$s_x = P\, n_x \qquad P = J F^{-1} \sigma F^{-T} = F^{-1} S$

$\downarrow$ traction

$$\begin{tikzcd}
\text{normal } n_y \arrow[rr, "\sigma"] & & \text{traction } s_y \\
n_x \arrow[rru, "S"] \arrow[rr, "P"'] & & s_x
\end{tikzcd}$$

In summary:

- Cauchy stress σ maps area in current configuration dA_y to traction in current configuration $\vec{s}_y$
- PK-I stress S maps area in reference configuration dA_x to traction in current configuration $\vec{s}_y$
- PK-II stress P maps area in reference configuration dA_x to traction in reference configuration $\vec{s}_x$

σ

$S = J\sigma F^{-T}$

$P = J F^{-1} \sigma F^{-T}$

Eulerian:

$$\frac{D}{Dt}\int_{B_t} \rho \, dV_y - \int_{\partial B_t} \sigma \cdot n \, dA_y = \int_{B_t} \rho b \, dA_y$$

$p = \rho v$

σ maps $\vec{dA_y} \to \vec{s}_y$

Strong form
1. PDE: $\frac{\partial p}{\partial t} - \operatorname{div} \sigma = \rho b$
2. Jump conditions: $[\sigma]\vec{n}_x = [p]\, n_x$

Balance of angular momentum

$\sigma = \sigma^T$

Lagrangian:

PK-I $$\int_{B_0} \frac{D p_0}{Dt} \, dV_x - \int_{\partial B_0} \boxed{S \cdot} \, n_x \, dA_x = \int_{B_0} \rho_0 b \, dA_x$$

S (PK-I)

$S = J\sigma F^{-1}$

maps $\vec{dA_x} \to s_y$

Strong form
1. PDE: $\frac{D p_0}{Dt} - \operatorname{Div} S = \rho_0 b$
2. Jump condition: $[S] \cdot \vec{n}_x = [p_0]$

Balance of angular momentum

$FS = S^T F^T$

S (non symmetric)

PK-II $$\int \frac{D p_0}{Dt} \, dV_x - \int_{\partial B_0} (FP) \cdot n_x = \int \rho_0 b \, dA_x$$

P PK-II

$P = J F^{-1} \sigma F^{-T} = F^{-1} S$

map $\vec{dA_x} \to \vec{s}_x$

Strong form
1. PDE: $\frac{D p_0}{Dt} - \operatorname{Div}(FP) = \rho_0 b$
2. Jump condition: $[FP] \cdot \vec{n}_x = [p_0]$

Balance of angular momentum

$P = P^T$

Lagrangian

Balance of energy:

$$\frac{D\mathcal{E}}{Dt} = \frac{D}{Dt}\int_{B_t} e\, dV_y = \int_{B_t} r_e^y\, dV_y - \int_{\partial B_t} f_e^y \cdot d\vec{A}_y$$

$\frac{D\mathcal{E}}{Dt}$: Scalar

$\int_{B_t} e\, dV_y$: scalar

e: volumetric energy density

r_e^y: scalar, energy source term

f_e^y: vector, outward spatial flux of energy

$$e = \underbrace{\underbrace{\tfrac{1}{2}\rho v \cdot v}_{\text{kinetic energy density}} + \underbrace{U(F, \dot{F})}_{\text{strain based energy density}}}_{\text{Mechanical}} + C_v T$$

$\tfrac{1}{2}\rho v \cdot v = \tfrac{1}{2} p \cdot v$: p covector, v vector

$\tilde{e}(F, T)$

$C_v T$: internal energy from temperature; C_v: volumetric capacity

$$+\tfrac{1}{2}(E \cdot D + H \cdot B) + \ldots$$

E: electric field (vector)
D: electric flux (covector)
H: Magnetic field (vector)
B: Magnetic flux

$$r_\varepsilon^y = \rho \vec{b} \cdot \vec{v} + Q$$

$\rho \vec{b}$: $\frac{\text{force}}{\text{volume}} \times \text{velocity} = \frac{\text{Power}}{\text{Volume}}$

Q ⟶ heat source

$+E \cdot J$ ⟶ Joule's heating

J: electric current density

$$f_e^y = -\sigma \cdot \vec{v} + Q - H$$

$$f_e^I = \underbrace{-\sigma \cdot \vec{v}}_{} + \underbrace{q}_{\text{outward heat flux}} + \underbrace{E \times H}_{\text{Poynting vector from EM.}} + \dots$$

spatial flux

σ: "force"/surface

$\vec{v}$: velocity,

power/surface

Ignore Mechanical & EM contributions

$$\left.\begin{aligned} e &= C_v T \\ r_e^y &= Q \\ f_e^y &= q \end{aligned}\right\} \longrightarrow \frac{\partial C_v T}{\partial t} + \operatorname{div} q = Q$$

no advection, only heat diffusion

eg $q = -K\nabla T \longrightarrow$ parabolic heat law

Energy balance

$$\frac{D}{Dt}\int_{B_t} e\, dV_y = \underbrace{\int_{B_t}\frac{\partial e}{\partial t}\, dV_y + \int_{\partial B_t} e\, v\cdot n\, dA_y} = \int_{B_t} r_e\, dV_y - \int_{\partial B_t} f_e^y \cdot n\, dA_y$$

Strong form

- $\frac{\partial e}{\partial t} + \mathrm{div}(F_e^y) = r_e$ PDE

$$F_e^y = \underbrace{f_e^y}_{\text{"diffusive"}} + \underbrace{e\vec{v}}_{\text{advection}}$$

- $[F_e^y]\cdot\vec{N}_y + [e]\cdot N_t = 0$ Jump law

e, r_e, f_e^y given above

Navier-Stokes equations:

1. Balance of mass (scalar) Continuity $\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho v) = \frac{D\rho}{Dt} + \rho\, \mathrm{div}\vec{v} = 0$
2. Balance of linear momentum (vector) $\frac{\partial p}{\partial t} + \mathrm{div}(p\otimes v - \sigma) = \rho b$
   (m is also used for p) $p = \rho\vec{v}$
3. Balance of energy $\frac{\partial e}{\partial t} + \mathrm{div}(ev - f_e^y) = r_e$
4. Balance of angular momentum $\sigma = \sigma^T$ e: volumetric energy density

+ Constitutive equation
  Fluids

Solids & Lagrangian representation

(2) Balance of linear momentum

(2) Balance of linear momentum

$$\frac{D p_0}{Dt} + \mathrm{Div}\cdot(S) = \frac{D p_0}{Dt} + (FP) = \rho_0 b$$

PK I $\quad S = J\sigma F^{-t} \quad P = JF^{-1}\sigma F^{-t}$

$p_0 = \rho_0 V$

(3) Energy is automatically satisfied if have only mechanical effects & linear momentum (2) is satisfied

(4) Balance of angular momentum $\quad P = P^T \ (\Longleftrightarrow \sigma = \sigma^T)$

$S = J\sigma F^{-t}$ if $H = O(\varepsilon)$ infinitesimal theory holds

$J \approx 1 + \mathrm{trace}(H) = O(\varepsilon) \quad F^{-t}_{\ldots} F_{\ldots} = 1 + O(\varepsilon) \quad S = \sigma + O(\varepsilon)$

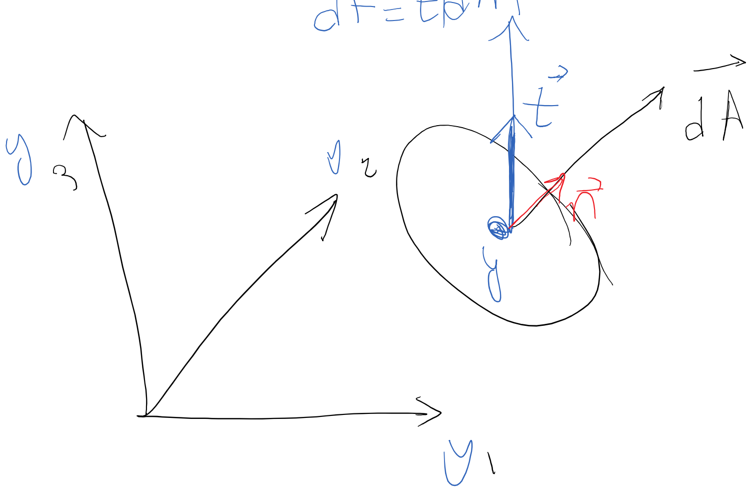
$P = \sigma + O(\varepsilon)$

all are the same within ε

(2) $\longrightarrow \quad \frac{D p_0}{Dt} + \mathrm{Div}\cdot\sigma = \rho_0 b \quad$ infinitesimal theory

Traction and stress tensor:

$\vec{dF} = \vec{t}\, dA$



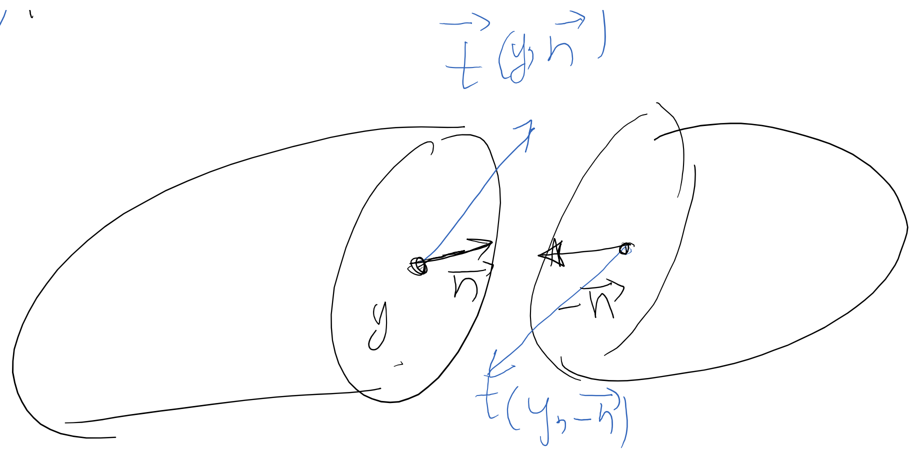
$t(y, \vec{n}) = \lim \frac{d\vec{F}_y}{dA}$

$dA \to 0$

$dF_y = \vec{t}\, dA$

$\vec{t}(y, \vec{n})$

Newton's law of action and reaction:



Why?

$t(y,-n) = -t(y,n)$

Balance of linear momentum

$$\frac{D}{Dt}\int_{P_t} \underbrace{\rho \vec{v}}_{dm}\, dV_y = \int_{P_t} \rho b \, dV_y - \int_{\partial P_t} \vec{t}\, dA_y$$

$\vec{t}$: traction vector

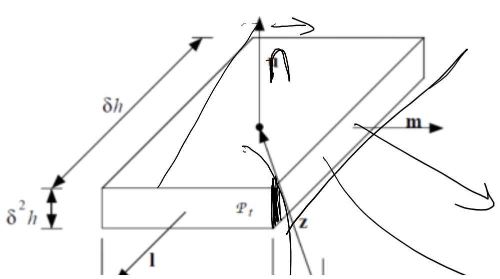
Using reduced transport

$$\int_{P_t} \rho \underbrace{\frac{D\vec{v}}{Dt}}_{\vec{a} \text{ acceleration}} dV_y = \int_{P_t} \rho \vec{b}\, dV_y - \int_{\partial P_t} \vec{t}\, dA_y$$

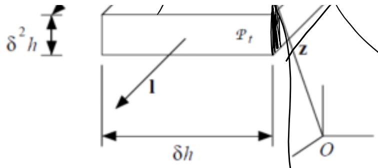
$$\boxed{\int_{P_t} \rho(\vec{a} - \vec{b})\, dV_y + \int_{\partial P_t} \vec{t}\, dA_y = 0}$$

Balance of angular momentum

P_t



Volume of $P_t = (\delta h^2)(\delta h)(\delta h) = \delta^4 h$



$$\text{Surface} = (\delta h^2)(\delta h) = \delta h^3 \qquad \delta^4 h$$

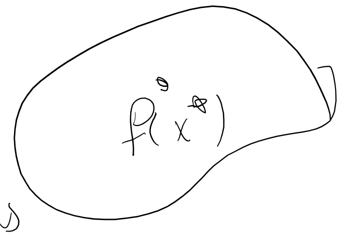
$$\text{surface} = (\delta h)(\delta h) = \delta h^2$$

is going to dominate as $\delta h \to 0$

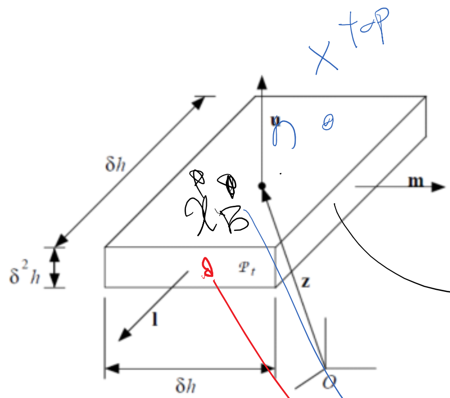
$$f(x^*)|D| = \int f(x)\, dV_x$$

mean theorem

continuous



$$\int_{P_t} \rho(\vec{a}-\vec{b})\, dV_y = \rho(\vec{a}-\vec{b})(x_b^*)|P_t| = \rho(\vec{a}-\vec{b})(x_b^*)\delta h^4$$



side integrals

$$\int_{\text{surface}} t\, dA_y = t(\text{some point})\, \delta h^3$$

$$\int t\, dA_y = \vec{t}(x^{top})\, \delta h^2$$

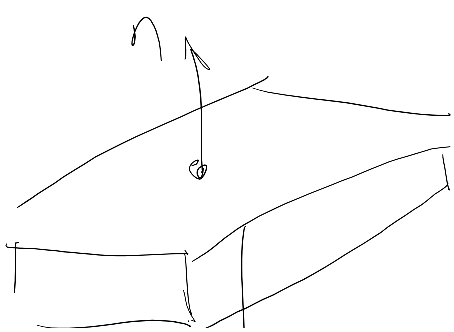
$$\int_{\partial B_{\text{bottom}}} \vec{t}\, dA_y = t(x^{\text{bottom}})\, \delta h^2$$

Add all & let $\delta h \to 0$ $\qquad \delta h^3, \delta h^4,$

$$\left\{ t(x^{top}) + t(x^{\text{bottom}}) \right\} \delta h^2 + \text{HOT} = 0$$

let $\delta h \to 0$

$$t(y, n) + \vec{t}(y, -n) = 0$$



$$\boxed{t(y, -n) = -t(y, n)}$$