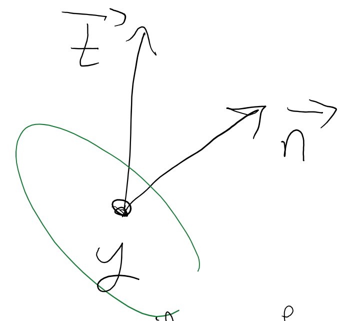
2019/11/11

Monday, November 11, 2019 11:33 AM

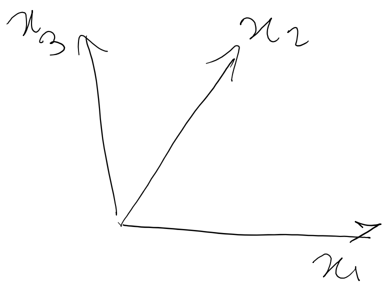
There exists a 2nd order tensor that delivers traction for any given direction n:

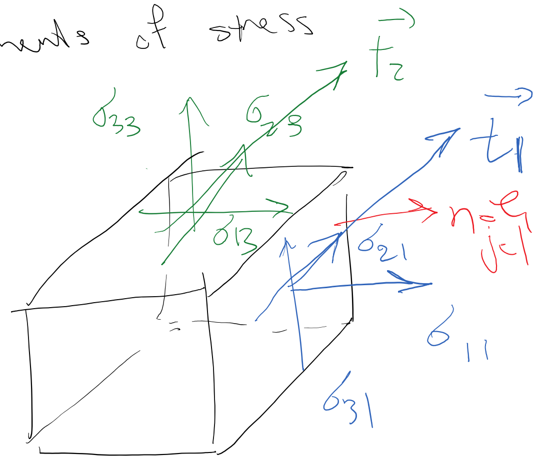
$$\vec{t}(y, \vec{n}) = \sigma(y) \cdot \vec{n}$$

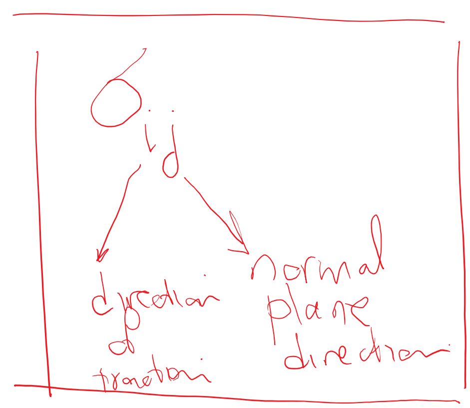


→ spatial configuration

What are the components of stress tensor?







Based on this definition:

1. Is σ a second order tensor?

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

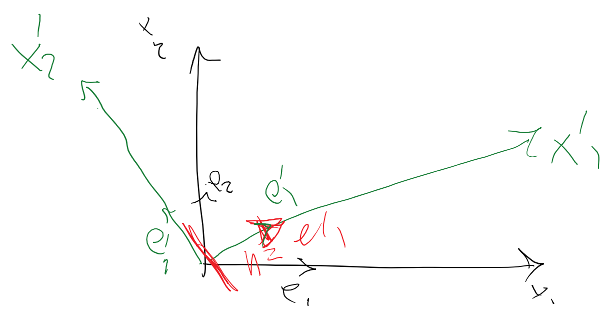
2. Does the relation $\vec{t} = \sigma \cdot n$ hold?

1. Since we define stress components for a given coordinate (definition is based on 1 coordinate system) we need to show that this 2 array matrix is in fact a second order tensor

$$Q = \begin{bmatrix} e'_1 \\ \hline e'_2 \\ \hline e'_3 \end{bmatrix} \rightarrow 3D$$

$$[\sigma'] = Q \sigma Q^T$$

$$\sigma'_{..} = Q_{..} Q_{..} \sigma_{..} ? \quad (\star)$$



$$\sigma'_{mn} = Q_{mi} Q_{nj} \sigma_{ij} \; ? \qquad (*)$$

We need to verify this:

In fact stress is a tensor and follows tensor transformation rules. The geometric representation of (*) is the Mohr circle.

To prove that it's a tensor we are going to use

$$\vec{t} = \sigma \cdot \vec{n}$$

Once we answer this question, it can be used in the proof that stress is a tensor.

why $\vec{t} = \sigma \cdot \vec{n}$?

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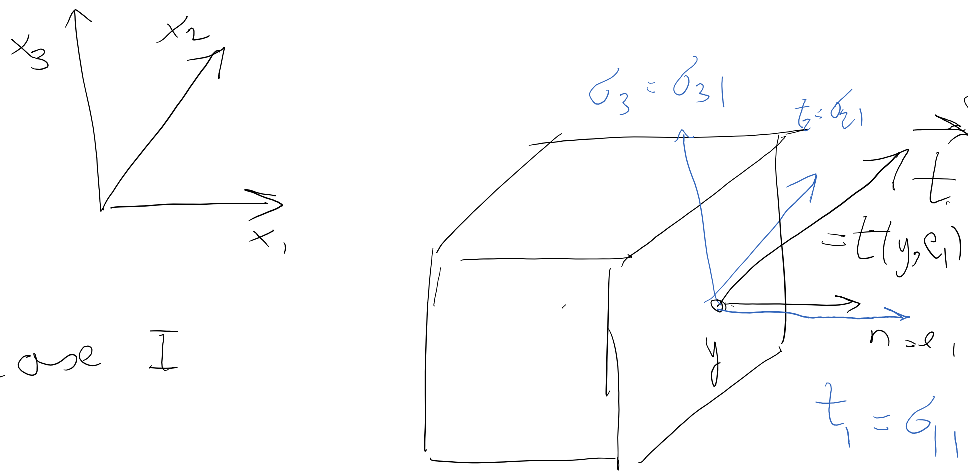
case 1 $n = (1,0,0),\ (0,1,0), \ldots$ two components are zero

case 2 $n = (n_1, n_2, 0) \ldots$ 1 component is zero

case 3 $n = (n_1, n_2, n_3)$ all n_i's are nonzero



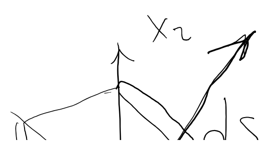
Case I

$$t = \begin{bmatrix} \sigma_{11} \\ \sigma_{21} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{31} & \sigma_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \checkmark$$

$n = e_1$ & likewise $n = e_2$ & e_3 this holds

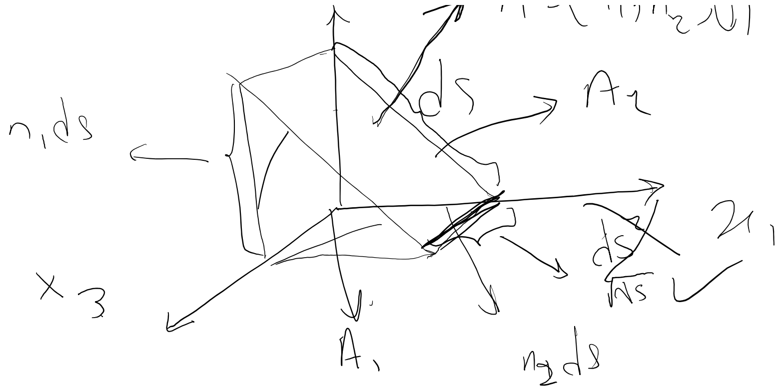
Case 2 One component of n is zero

e.g. $n = (n_1, n_2, 0)$



e.g. $n = (n_1, n_2, 0)$

balance of linear momentum
get rid of volume contributions
& surface ones on A_1 & A_2

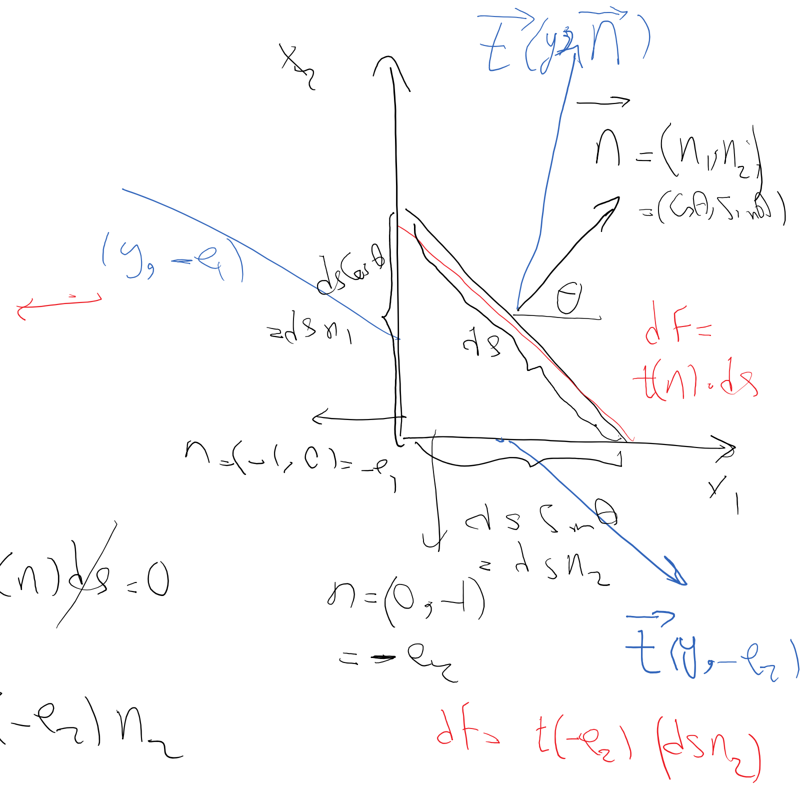


Sloppy proof:

$\vec{dF} = \vec{t}(\Delta A)$ (area)

$dF = \vec{t}(ds\, n_1)$

$t(-e_1)\, ds\, n_1$



$\sum F = 0$

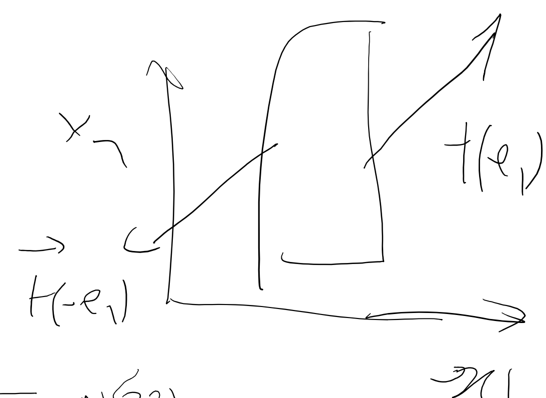
$t(e_1) n_1 ds + t(-e_2) ds\, n_2 + t(n) ds = 0$

$t(n) = -t(-e_1) n_1 - t(-e_2) n_2$

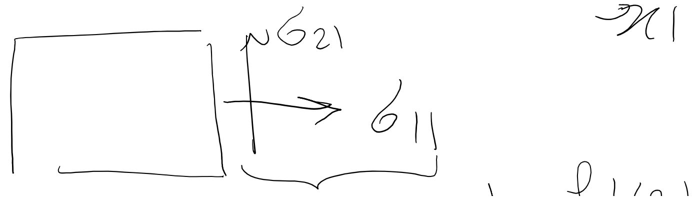
$t(e_1) n_1 + t(e_2) n_2$

$= \begin{bmatrix} \sigma_{11} \\ \sigma_{21} \end{bmatrix} n_1 + \begin{bmatrix} \sigma_{12} \\ \sigma_{22} \end{bmatrix} n_2$

$= \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$ ✓



$t(n) = \sigma \cdot n$



$t(n) = \sigma \cdot n$

$|\_\_\_\_\_||T \to \sigma_{11}|$ components of $t(e_1)$ by definition

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Case 3 : $n_1 \neq 0 \quad n_2 \neq n_3 = 0$

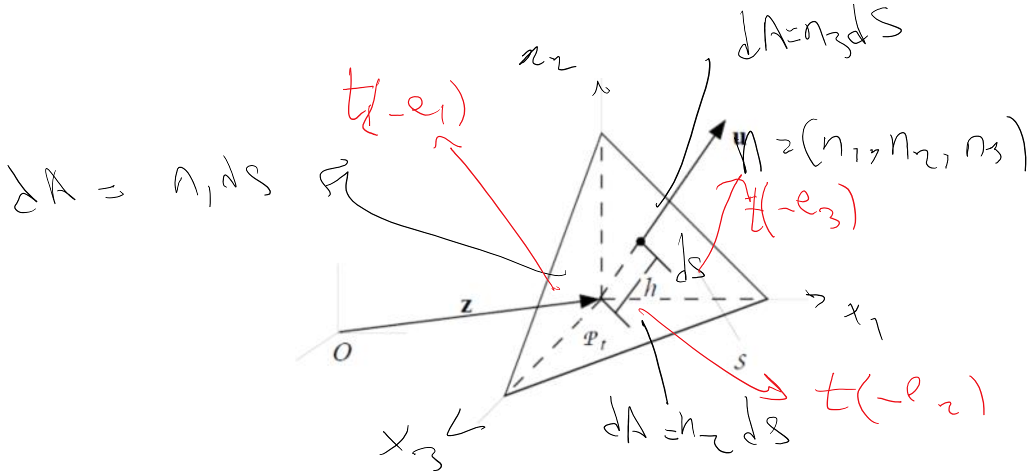
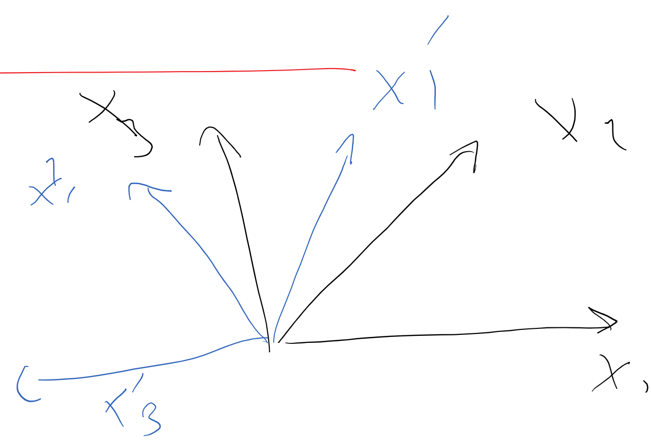


Figure 3.5: Tetrahedron of altitude h located at $\mathbf{z} \in \mathcal{B}_t$

Follow the same line of proof as we did for previous case to show

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \sigma \cdot n$$

$$t = \sigma \cdot n$$



choose $n = e'_1$
e'_2
e'_3

and compute $t(e'_1)$ $t(e'_2)$ $t(e'_3)$ in (e_1, e_2, e_3) coordinate system ($\sigma e'_1$, $\sigma e'_2$, $\sigma e'_3$) & express them in (e'_1, e'_2, e'_3) system . . .

$\longrightarrow$ this shows $\sigma' = Q \sigma Q^T$

→ this shows $\sigma = Q\,\sigma\,Q$

shows σ is a 2nd tensors following coordinate transformation rules

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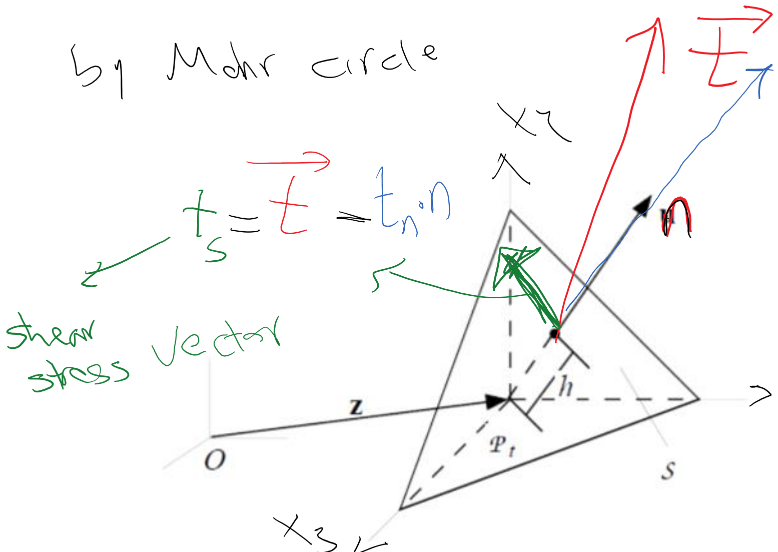
Mohr circle: we did it for 2D in class

2nd order transformation geometric rep. for symmetric tensor

In 3D we also can show tensor transformation

$$[\sigma'] = Q[\sigma]Q^T$$

by Mohr circle



$t_n = \vec{t} \cdot n = (\sigma n) \cdot n$
$= n \cdot (\sigma n)$

scalar normal stress

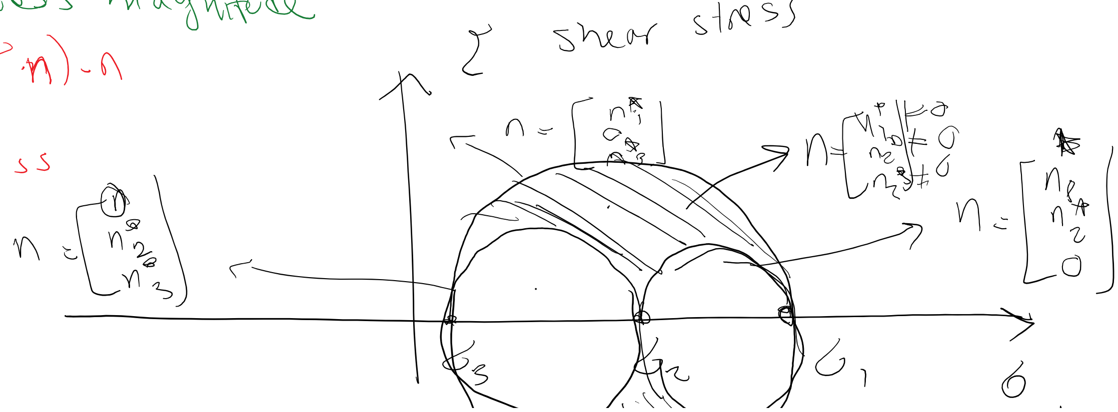
Normal traction vector $= (t_n)\vec{n}$

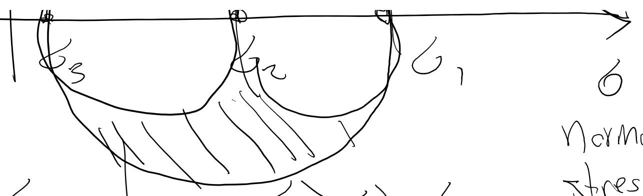
$\tau(n) = |t_s| = |\vec{t} - t_n \cdot n|$

shear stress magnitude

$\sigma(n) = (\sigma \cdot n) \cdot n$

normal stress

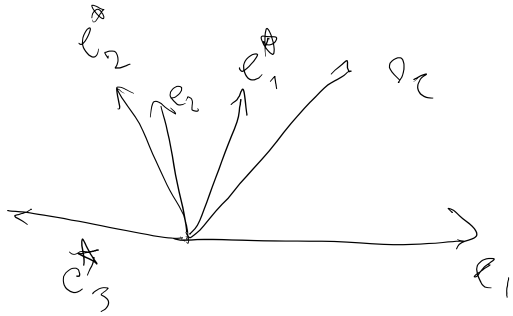




$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots \\ & & \\ & & \end{bmatrix} \rightarrow [\sigma^*] = \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{bmatrix}$ principal stresses

$\sigma_1 \geq \sigma_2 \geq \sigma_3$

It's symmetric (next)



see section

1.15.2 for proving this.

3D simply use $[\sigma'] = Q[\sigma]Q^T$

Why stress tensor is symmetric?

Axiom 5 (Principle of Balance of Angular Momentum) *$\forall\, \mathcal{P} \subset \mathcal{B}$ and $\forall\, t \in [t_0, \infty)$,*

$$\int_{\partial \mathcal{P}_t} \left(\mathbf{y} - \overset{0}{\mathbf{x}}\right) \times \mathbf{t}_{\mathbf{n}(\mathbf{y},t)}(\mathbf{y},t)\, dA_y + \int_{\mathcal{P}_t} \left(\mathbf{y} - \overset{0}{\mathbf{x}}\right) \times \mathbf{b}(\mathbf{y},t)\rho(\mathbf{y},t)\, dV_y$$
$$= \frac{d}{dt}\int_{\mathcal{P}_t} \left(\mathbf{y} - \overset{0}{\mathbf{x}}\right) \times \hat{\mathbf{v}}(\mathbf{y},t)\rho(\mathbf{y},t)\, dV_y,$$

where $\overset{0}{\mathbf{x}}$ is the position vector of an arbitrary fixed point of $\mathcal{E}$. That is, the total moment of the forces acting on any part of a body about an arbitrary fixed point at any time t is balanced by the instantaneous time rate of change of the part's angular momentum about the same point.

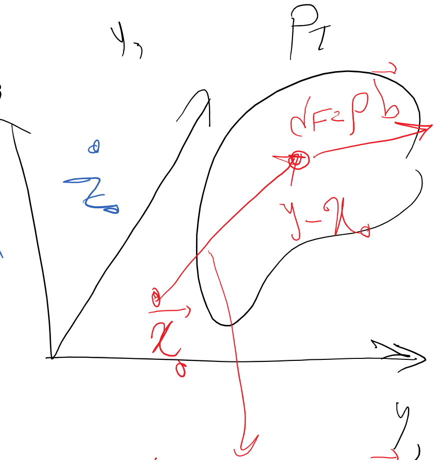
div theorem (★) $\frac{d}{dt}$

It holds wrt x_0 & balance of linear momentum holds

(★) holds for $\overset{0}{x} \to z_0$

Exercise 84 *Assume that the forces acting on a body and its velocity satisfy the Principle of Balance of Linear Momentum. Show that under this assumption, if the angular momentum about any one fixed point is balanced, then it is balanced about all fixed points in $\mathcal{E}$ (not necessarily in the body). [Hint: Begin by demonstrating that, for any two fixed points $\overset{0}{\mathbf{x}}$ and $\overset{0}{\mathbf{z}}$,*

$$\int_{\mathcal{P}_t} \left(\mathbf{y} - \overset{0}{\mathbf{z}}\right) \times \hat{\mathbf{v}}(\mathbf{y},t)\rho(\mathbf{y},t)\, dV_y = \int_{\mathcal{P}_t} \left(\mathbf{y} - \overset{0}{\mathbf{x}}\right) \times \hat{\mathbf{v}}(\mathbf{y},t)\rho(\mathbf{y},t)\, dV_y + \int_{\mathcal{P}_t} \left(\overset{0}{\mathbf{x}} - \overset{0}{\mathbf{z}}\right) \times \hat{\mathbf{v}}(\mathbf{y},t)\rho(\mathbf{y},t)\, dV_y.$$



$\sum M_{\text{around } x_0} = 0$

after $\frac{d}{dt} \rightarrow$ inside

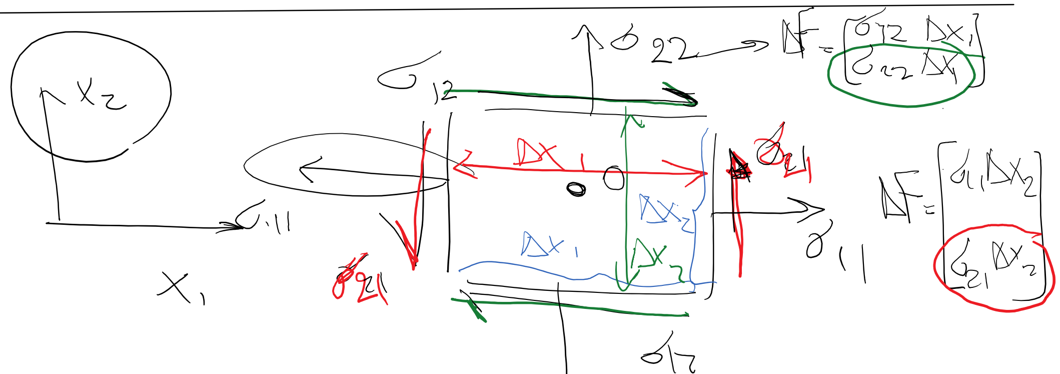
∂B_t intergral $\rightarrow B_t$ integral (div theorem)

& localization

$\epsilon_{ijk} \underbrace{T_{kj}}_{\sigma_{ji}} = 0 \implies \underbrace{T_{kj} = T_{jk}}_{\text{same}\ \sigma_{ji} = \sigma_{ij}}$

σ_{kj} — same thing — $\sigma_{kj} = \sigma_{jk}$

Sloppy proof:



$\sum M_o = 0$

$$(\sigma_{21}\Delta x_2)\Delta x_1 - (\sigma_{12}\Delta x_1)\Delta x_2 = 0 \qquad \sigma_{22}$$

moment from σ_{21}'s

$$\sigma_{21} - \sigma_{12} = 0$$

sloppy way of showing symmetry of σ

Constitutive equations (last chapter):

infinitesimal theory

1. $G = \frac{1}{2}(C - I)$ Kinematics — $E = \frac{1}{2}(\nabla u + \nabla u^T)$

2. $\rho_0 \frac{DV}{Dt} - \mathrm{Div}\, P = \rho_0 b$ (P → PK-I) Forces / balance laws — $\rho_0 \frac{DV}{Dt} - \mathrm{Div}\, \sigma = \rho_0 b$

3. $P = g(G, \dots)$ relates kinematic quantities to force like quantities — $\sigma = g(E)$

Constitutive eqn $\quad G, \dots$

Definition 108 *A body* $\overset{0}{\mathcal{B}}$ *is* (Cauchy) elastic *if* $\exists$ *a function*

$$\mathbf{G} : \mathrm{Lin}\ \mathcal{V}^{+} \times \overset{0}{\mathcal{B}} \rightarrow \mathrm{Sym}$$

$\ni$ *the referential Cauchy stress field corresponding to the motion* $\{\mathbf{f}(\cdot, t)\}$ *is given by*

$$\tilde{\mathbf{T}}(\mathbf{x}, t) = \mathbf{G}(\mathbf{F}(\mathbf{x}, t), \mathbf{x}), \ (\mathbf{x}, t) \in \overset{0}{\mathcal{B}} \times [t_0, \infty).$$

Elastic material

$F \rightarrow \sigma \quad (T)$

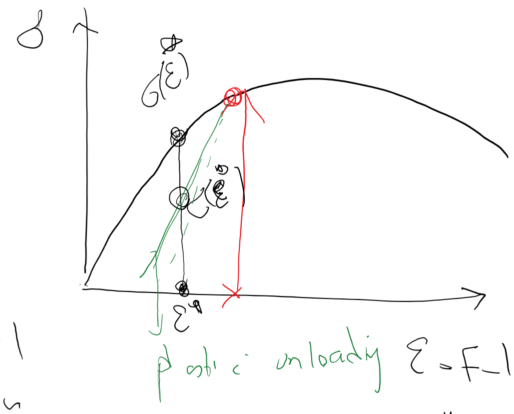
∋ *the referential Cauchy stress field corresponding to the motion* $\{\mathbf{f}(\cdot,t)\}$ *is given by*

$$\tilde{\mathbf{T}}(\mathbf{x},t) = \mathbf{G}(\mathbf{F}(\mathbf{x},t),\mathbf{x}),\ (\mathbf{x},t) \in \overset{0}{\mathcal{B}} \times [t_0,\infty).$$

and the corresponding spatial Cauchy stress field has the smoothness property

$$\mathbf{T}(\mathbf{y},t) := \tilde{\mathbf{T}}(\mathbf{f}^{-1}(\mathbf{y},t),t) \in C^1(\mathcal{B}_t).$$

The function $\mathbf{G}$ *is called the* elastic response function, *and the above equation is called the* elastic constitutive equation.

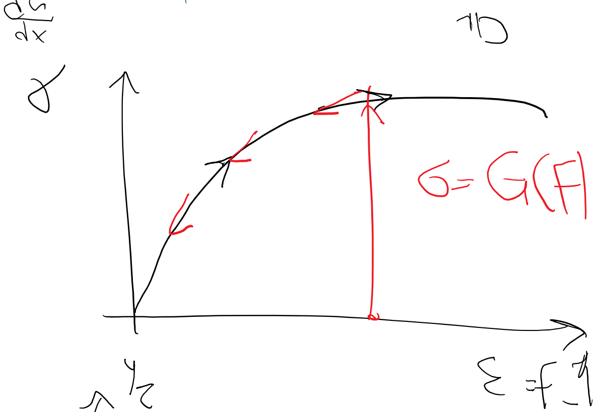


1D $\varepsilon = F - 1 = \frac{dy}{dx} - 1 = \frac{du}{dx}$

$\tilde{T}(x) = G(F, X)$

Cauchy stress (σ)

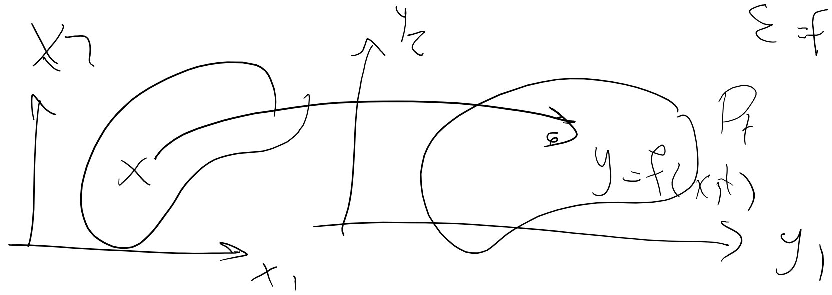
$F = \frac{dy}{dX}$



$T(y,t)$

$= T(f^{-1}(x,t),t)$

$= \tilde{T}(x,t)$



$\tilde{T} = G(F, X)$

can model inhomogeneous response

Next sessions

Objectivity ⟶ $T = G(F, X)$

$G(F^t F, X)$

$$G(\underbrace{F^{t}f}_{C}, X)$$