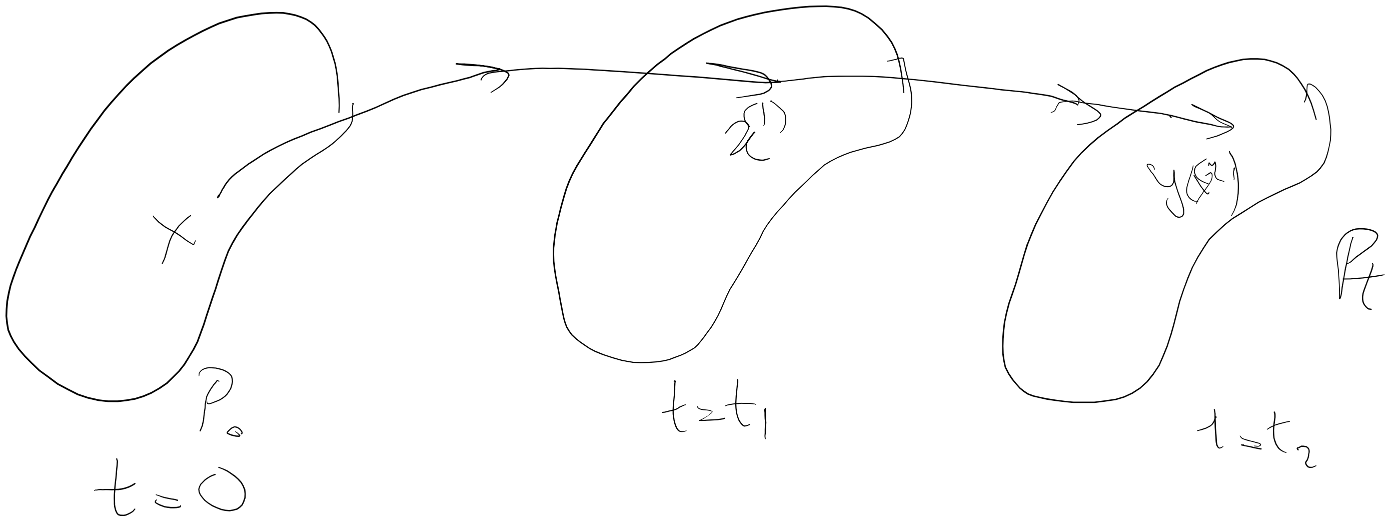
2019/11/13

Wednesday, November 13, 2019 11:30 AM

$T = G(F, X)$

σ

What is the importance of reference configuration:



what to write constitutive eqn once with t_0 being reference, once t_1

$y = x^{(1)} = f(X, t_1)$ $\quad (y = f(X, t_1))$

$\longrightarrow \quad X = f^{-1}(x^{(1)}, t_1)$

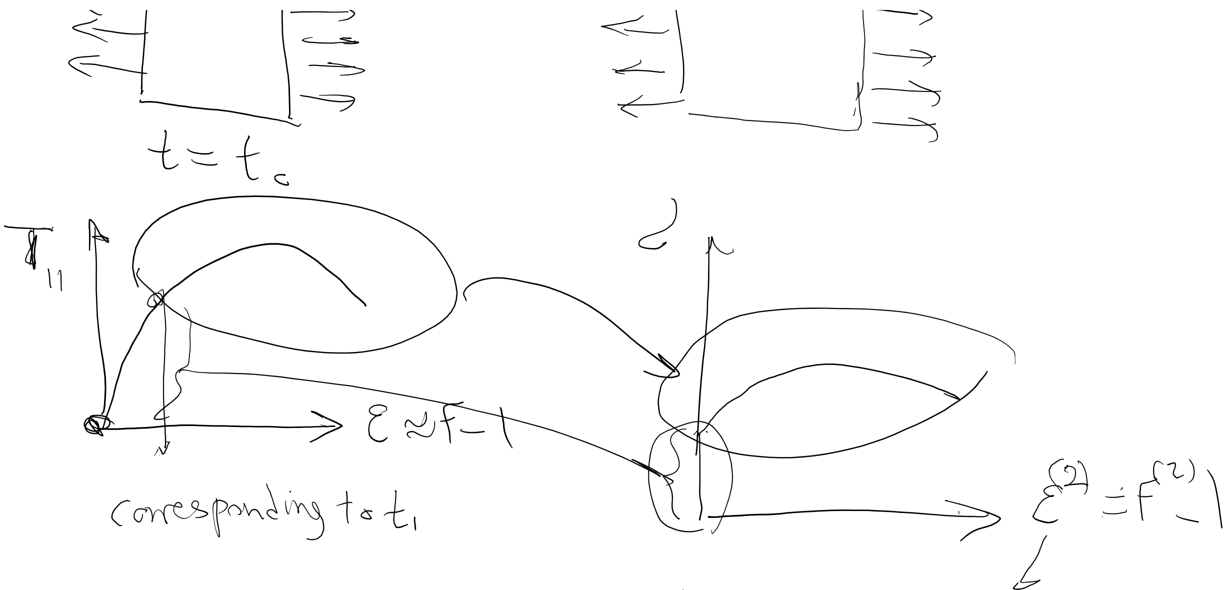
y at time t_2 : $\quad y = f(X, t_2) = f(f^{-1}(x^{(1)}, t_1), t_2 - t_1 + t_1)$

$= f^{(2)}(x^{(1)}, t')$

$t' = t_2 - t_1$

having deformation $y = f(X, t)$ with respect to t_0 we can write

$y = f^{(2)}(x^{(1)}, t')$ which is the deformation w r t t_1



$G(F, x)$

$G^{(2)}(F^{(2)}, x)$

strain w.r.t. t_1

If we have the option, we want to choose an initial (reference) configuration where stress is zero for zero strain (F = I)

Remark 54 *It is commonly (but by no means universally) assumed that the reference configuration represents a* natural state of the body $\ni$ *the stress field vanishes (i.e., there is no* initial stress*). This assumption of course requires that*

$$\mathbf{G}(\mathbf{I}, \mathbf{x}) = \mathbf{0} \;\forall\;\; \mathbf{x} \in \overset{0}{\mathcal{B}}\,.$$

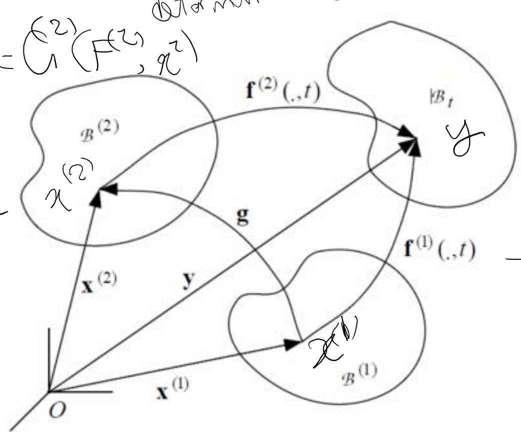
Although this assumption is not always appropriate, we shall nonetheless adopt it for reasons of simplicity from here on.

$G(F = I, x) = 0$

How to relate constitutive equations from two different references:

$\tilde{T}(x^{(2)}) = G^{(2)}(F^{(2)}, x^{(2)})$ deformation (2)

$F^{(2)} = \nabla y_{/x^{(2)}}$



→ deformation 1

$\tilde{T}(x^{(1)}) = G^{(1)}(F^{(1)}, x^{(1)})$

$F^{(1)} = \nabla y_{/x^{(1)}}$

Relate $G^{(1)}$ & $G^{(2)}$: even though $G^{(1)}$ & $G^{(2)}$ are different, once we characterize one (e.g. experiments) we can express the other one

$$g: X^{(1)} \longrightarrow X^{(2)} \qquad g(X^{(1)}) = X^{(2)}$$

$$F^{(1)} = \nabla_{X^{(1)}} y$$

$$F^{(1)}_{ij} = \frac{\partial y_i}{\partial X^{(1)}_j} = \frac{\partial y_i}{\partial X^{(2)}_k} \frac{\partial X^{(2)}_k}{\partial X^{(1)}_j} = F^{(2)}_{ik} \frac{\partial g_k(X^{(1)})}{\partial X^{(1)}_j} = F^{(2)}_{ik} \nabla g_{kj}$$

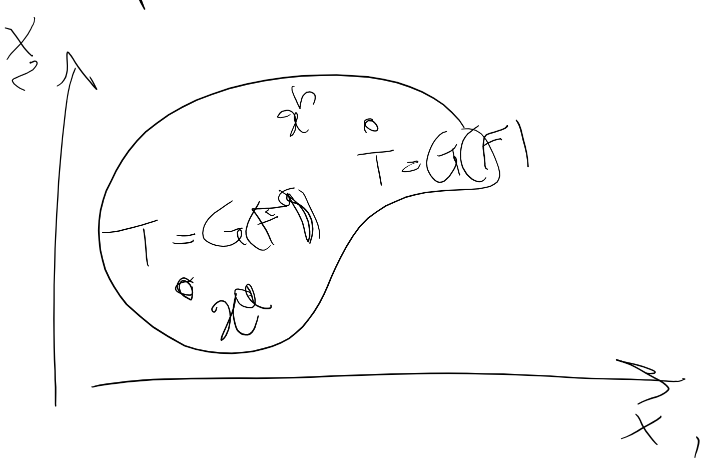
$$\boxed{\begin{aligned} F^{(1)} &= F^{(2)} \nabla g \\ \tilde{T} &= G^{(1)}(F^{(1)}, X^{1}) \\ &= G^{(2)}(F^{(2)}, X^{2}) \end{aligned}} \longrightarrow \boxed{\begin{aligned} \tilde{T} &= G^{(1)}(F^{(2)} \nabla g, g^{-1}(X^{2})) \\ &= G^{(2)}(F^{(2)}, X^{(2)}) \end{aligned}}$$

$G^{(1)}(F^{(1)}, X^{1})$ is known $\longrightarrow$ we obtain

$$\boxed{G^{(2)}(F^{(2)}, X^{(2)}) = G^{(1)}(F^{(2)} \nabla g, g^{-1}(X^{(2)}))}$$

$T = G(F, \underline{X})$ is homogeneous if it does not depend on X.

$$T = G(F)$$



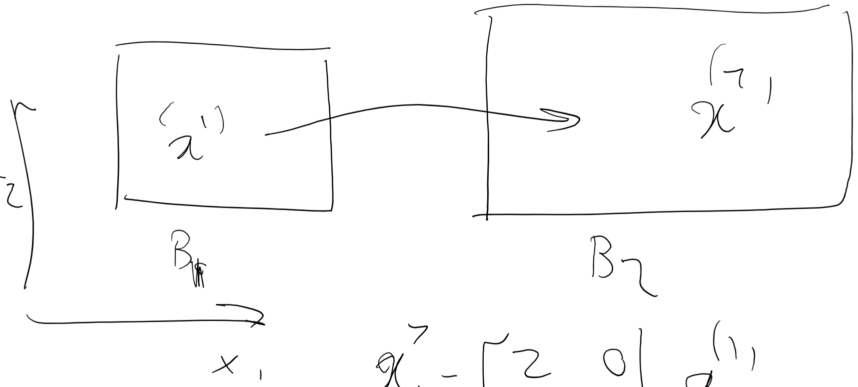
If the map between two initial states is homogeneous, and the material is homogeneous w.r.t. one initial state -> it is homogenous w.r.t. the other one.

$x^{(2)} = g(x^{(1)})$

∇g is constant

$$x^{(2)} = \underset{\sim}{A}\, x^{(1)}$$

constant matrix

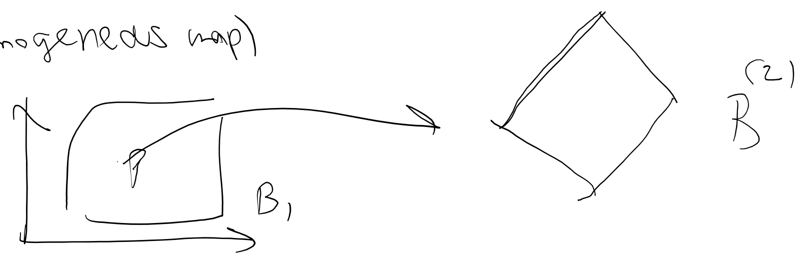


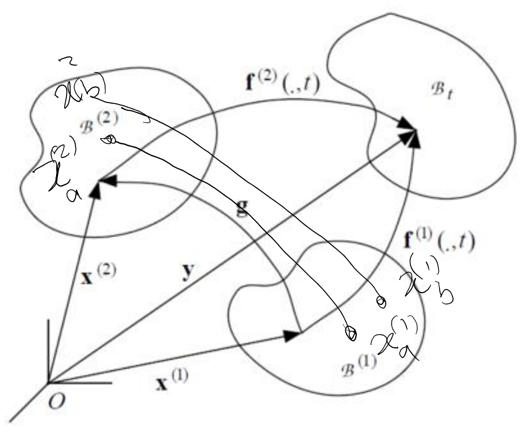
$$x^{2} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} x^{(1)}$$

how about

$$x^{(2)} = Q\, x^{(1)}$$

↳ rotation (homogeneous map)





Assumptions:

- $G^{(1)}$ is homogeneous

$\forall F$ & $x_a^{(1)}, X_b^{(1)}$: $\quad G^{(1)}(F, x_a^{(1)}) = G^{(1)}(F, x_b^{(1)})$ (1)

- g is homogeneous $G = \nabla g$ is constant (2)

Want to show $\quad G^{(2)}(F, X_a^{(2)}) = G^{(2)}(F, X_b^{(2)})$

$G^{(1)}(F\nabla g(X_a^{1}), X_a^{1}) \quad\quad \| \quad\quad G^{(1)}(F\nabla g(X_b^{1}), X_b^{1})$

(2) constant G $\quad\quad G$

FG the same FG

FG the same FG

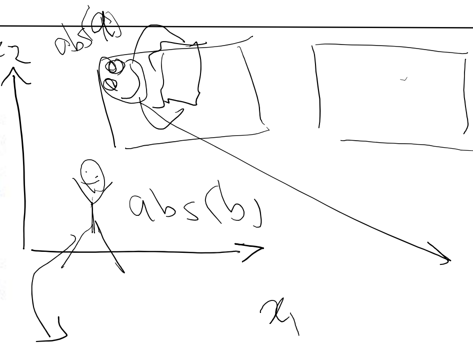
property 1 $G^{(1)}(FG, x'_a) = G^{(1)}(FG, x'_b)$

$G^{(2)}(F, x^{(2)}_a) = G^{(2)}(F, x^{(2)}_b)$

G^2 is also homogeneous

So if material is homogeneous w.r.t 1 state (1) it's homogeneous with respect to another state (2) if (2) is a homogeneous map of (1) ($x^{(2)} = g(x^{(1)})$ $\nabla g = G$ constant)

4.3 Principle of Material Frame-Indifference



This section explores the notion that material response is invariant under (indifferent to) superposed rigid motions and shifts in the origin of the time scale. Only invariance under superposed rigid motion is relavent in the context of elasticity theory which does not include memory effects. We begin with the notion of equivalent motions.

Definition 110 *Two motions of a body,* $\{\mathbf{f}(\cdot,t)\}$ *and* $\left\{\overset{*}{\mathbf{f}}(\cdot,t)\right\}$*, are equivalent w.r.t. material response if they differ by a rigid deformation for each* $t \in [t_0, \infty)$*; i.e.,* $\exists$ *functions* $\mathbf{c} : [t_0,\infty) \to \mathcal{V}$ *and* $\mathbf{Q} : [t_0,\infty) \to \text{Orth } \mathcal{V}^+ \ni$

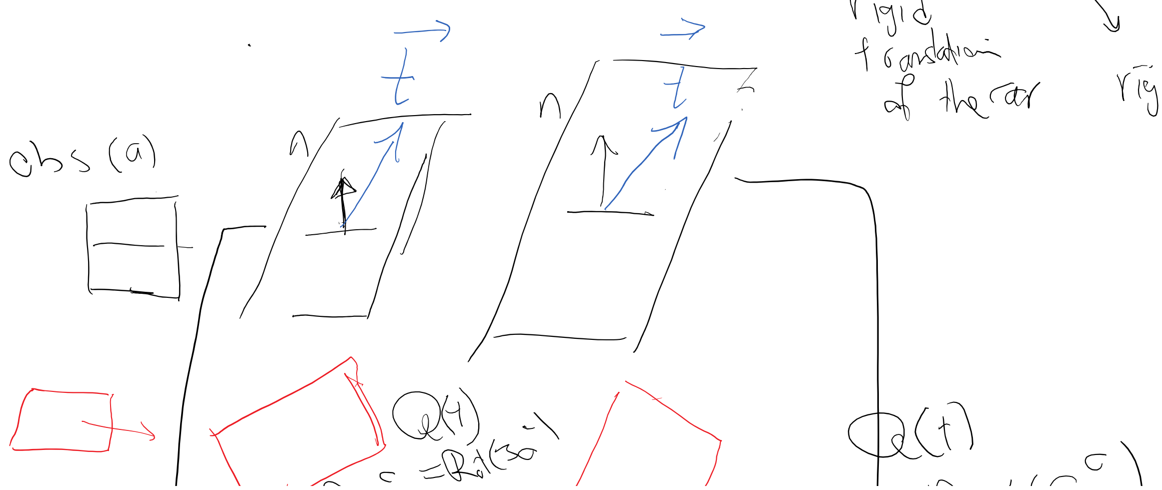
$$\overset{*}{\mathbf{f}}(\mathbf{x},t) = \mathbf{c}(t) + \mathbf{Q}(t)\mathbf{f}(\mathbf{x},t) \quad \forall\, (\mathbf{x},t) \in \overset{0}{\mathcal{B}} \times [t_0,\infty).$$

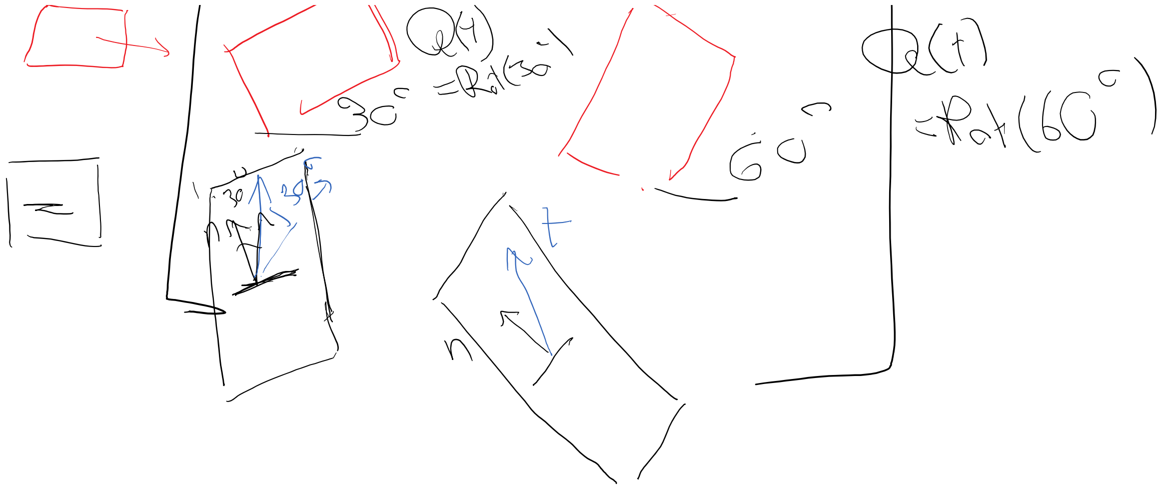
$y = f(x,t)$

$y = c(t) + Q(t) f(x,t)$

rigid translation of the car

rigid rotation of the car





Objectivity would require both normal vector and traction on arbitrary point and plane of interest be both rotated by Q(t) ob served by observer (b) relative to what observer (a) sees.

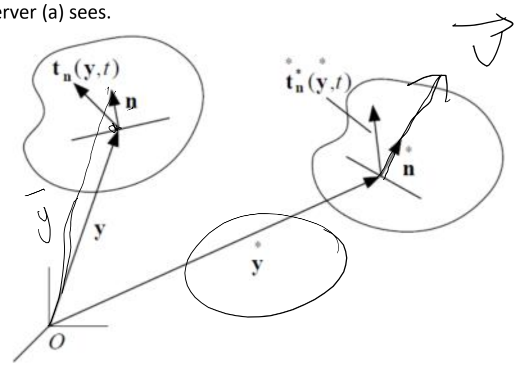


Figure 4.2: Surface tractions from equivalent motions

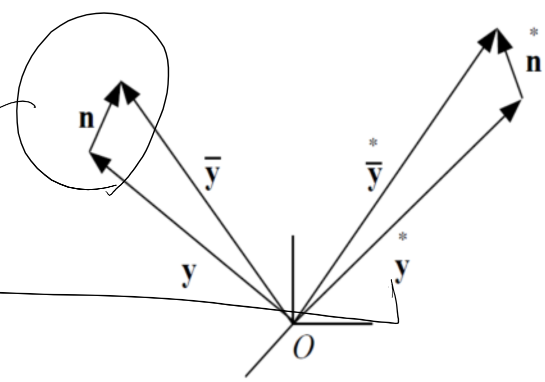
$$y^* = c(t) + Q(t)\, y(x,t)$$

y^*: obs (b); $c(t)$: rigid translation; $Q(t)$: rigid rotation

$\bar{y} - y = n$ size 1 & normal to the plane

$$y^* = c(t) + Q(t)\, y$$

$$\bar{y}^* = c(t) + Q(t)\, \bar{y}$$



$$\underbrace{\bar{y}^* - y^*}_{v} = Q(t)(\bar{y} - y)$$

$$= Q(t)\, n$$

v is in fact n^*

$$v \cdot v = (Q(t)\, n) \cdot Q(t)\, n$$

also unit size

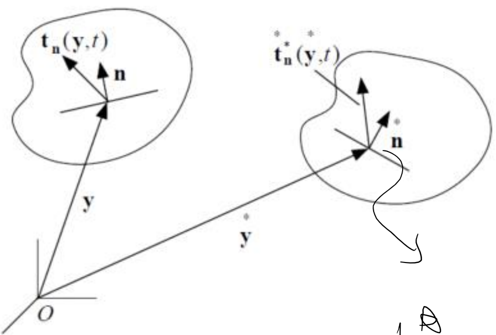
$$(\underbrace{Q^T(t)Q(t)}_{I}\, n) \cdot n$$

$n \cdot n = 1$

So $n^* = Qn$

So $n^* = Qn$

Objectivity requires $t^*_{n^*} = Q t_n$



$$t_n = T n \qquad t^*_{n^*} = T^* n^*$$

$$Q t_n = T^*(Qn)$$

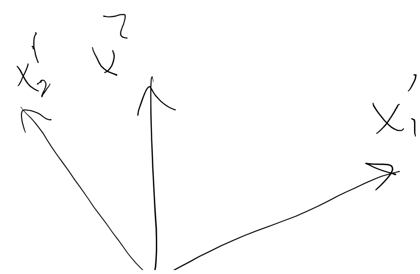
$$QTn = T^* Q n \quad \text{for all } n$$

$$Tn = (Q^{T} T^* Q)\, n \qquad \forall \vec{n}$$

$$\Rightarrow \quad T = Q^{-1} T^* Q \quad \Longrightarrow$$

$$\boxed{T^* = QTQ^t}$$

similar to



objectivity requires this

objectivity requires this

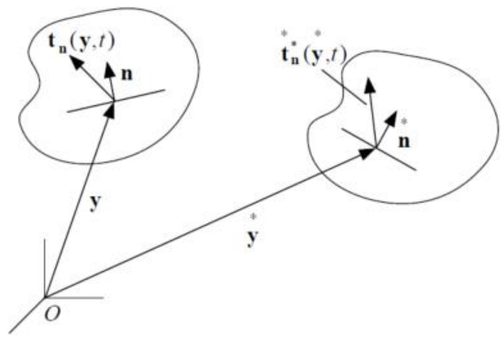


Figure 4.2: Surface tractions from equivalent motions

$$[T'] = Q[T]Q^T$$

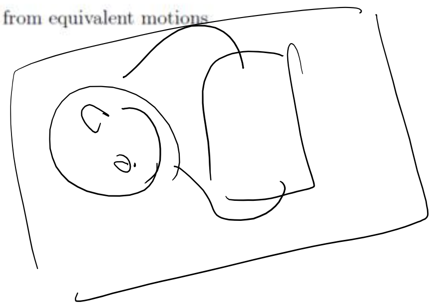
components

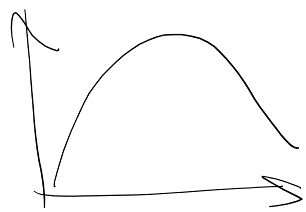
non-prime system

look the same thing

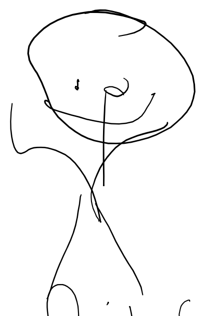
What is the use of this?

$$T = G(F)$$





$$\overset{*}{T} = G(\overset{*}{F})$$



$$\overset{*}{F}_{ij} = \frac{\partial \overset{*}{y}_i}{\partial x_j} = \frac{\partial (c_i(t) + Q_{ik}(t)\, y_k(x,t))}{\partial x_j} \qquad \overset{*}{y} = c(t) + Q(t)\, y$$

$$= Q_{ik}(t) \frac{\partial y_k}{\partial x_j} = Q_{ik} F_{kj}$$

$$\overset{*}{F} = QF$$

$T = G(F)$ kid

$\overset{*}{T} = G(\overset{*}{F})$ outside observer

$$T^{*} = Q T Q^{t} \quad \text{objectivity}$$

$$F^{*} = Q F$$

$$Q T Q^{t} = G(QF)$$

$$\boxed{Q G(F) Q^{t} = G(QF)}$$