2019/11/15

Friday, November 15, 2019 11:40 AM

Last time from objectivity we obtained:

$$\boxed{Q G(F) Q^t = G(QF)}$$

$F = RU$

$$G(\underbrace{QR}F) = Q\,G(F)Q^t$$

choose $Q = R^t$

$$G(\underbrace{R^tR}_{I}\underbrace{U}_{F}) = R^t G(F)(R^t)^t = R^t G(F) R$$

$$\longrightarrow \quad G(U) = R^t G(F) R \rightarrow \quad ⓐ \boxed{G(F) = R\,\underbrace{G(U, x)}\,R^t}$$

$F = RU$

$$R = FU^{-1} \xrightarrow{ⓐ} G(F) = F\underbrace{U^{-1} G(U, x)\, U^{-t}}F^t$$

define $\hat{G}(U, x) = U^{-1} G(U, x) U^{-t}$

$$\boxed{G(F) = F\hat{G}(U, x)F^t} \quad ⓑ$$

$U = \sqrt{C}$ $\qquad G(F) = F\,\underbrace{\hat{G}(\sqrt{C}, x)}\,F^t$

$$\bar{G}(C, x) = \hat{G}(\sqrt{C}, x)$$

$$\Rightarrow \boxed{G(F) = F\bar{G}(C, x)F^t} \quad ⓒ$$

So by considering objectivity a constitutive equation in the form $\tilde{T}(x, t) = G(F, x)$ takes one of the following forms:

Theorem 173 *If the elastic constituitive equation*

$$\tilde{\mathbf{T}}(\mathbf{x},t) = \mathbf{G}(\mathbf{F}(\mathbf{x},t),\mathbf{x}) \tag{4.1}$$

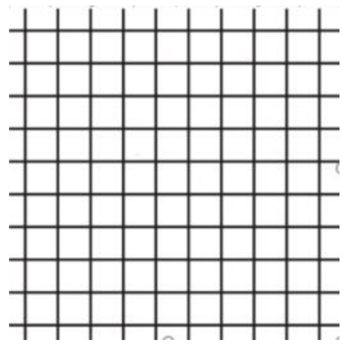
is consistent with the Principle of Material Frame-Indifference, then it can be written in any of the following reduced forms*:*

$$\begin{aligned}
\tilde{\mathbf{T}}(\mathbf{x},t) &= \mathbf{R}(\mathbf{x},t)\mathbf{G}(\mathbf{U}(\mathbf{x},t),\mathbf{x})\mathbf{R}^t(\mathbf{x},t); && (4.2)\\
\tilde{\mathbf{T}}(\mathbf{x},t) &= \mathbf{F}(\mathbf{x},t)\hat{\mathbf{G}}(\mathbf{U}(\mathbf{x},t),\mathbf{x})\mathbf{F}^t(\mathbf{x},t); && (4.3)\\
\tilde{\mathbf{T}}(\mathbf{x},t) &= \mathbf{F}(\mathbf{x},t)\bar{\mathbf{G}}(\mathbf{C}(\mathbf{x},t),\mathbf{x})\mathbf{F}^t(\mathbf{x},t); && (4.4)
\end{aligned}$$

where $\hat{\mathbf{G}} : \mathrm{Psym} \times \overset{0}{\mathcal{B}} \to \mathrm{Sym}$ *and* $\bar{\mathbf{G}} : \mathrm{Psym} \times \overset{0}{\mathcal{B}} \to \mathrm{Sym}$.

4.4 Material symmetry; Isotropy

From geometric perspective, any multiple of 90 degree rotation, results in the same material.



A transformation of an object that leaves some properties of the object invariant is called a symmetry transformation.

How about a constitutive equation?

We will show that under infinitesimal deformation for linear material:

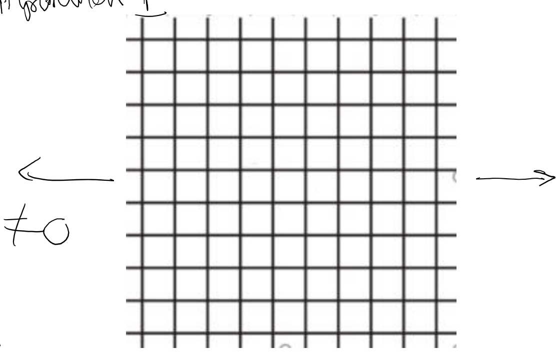
$$\underbrace{T}_{\text{stress}} = C_{\text{elasticity tensor}} \varepsilon$$

$T_{11} = C_{1111} \epsilon_{11}$ if all other strains are zero

configuration 1

$T^{(1)}_{11} = C^{(1)}_{1111} \epsilon^{(1)}_{11}$

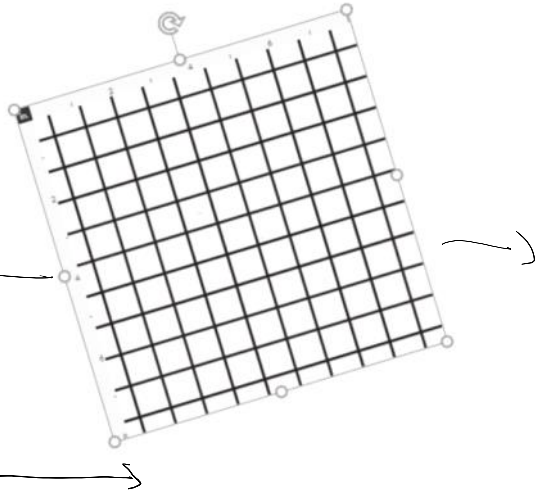
$\varepsilon^{①}$



$\varepsilon^{①} = \epsilon^{(1)}_{11} \neq 0$

$$T^{(2)}_{11} = C^{(2)}_{1111}\, \varepsilon^{(2)}_{11}$$

$$\varepsilon^{(2)}_{11} \to \varepsilon^{0}$$



$\varepsilon_{11} = \varepsilon^{0}$

$$C^{(1)}_{1111} \neq C^{(2)}_{1111}$$

in general

The constitutive eqn is not invariant w.r.t 30° rotation

It's clear that const. equation will be invariant w.r.t any multiple of 90° rotation.

90° rotations belong to symmetry group for constitutive equation

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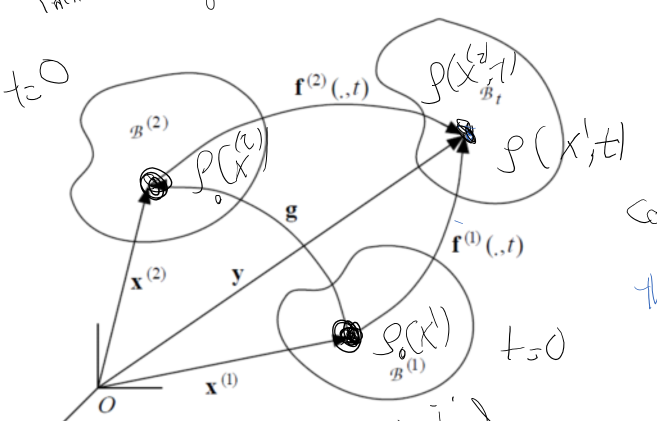
Last time we obtained the relation between G's starting from two different "initial configurations":

initial configuration (2)



$$x^{(2)} = g(X^{(1)})$$

density relation $\quad \dfrac{\rho}{\rho_0} = \quad J = \det F$

configuration (1).

$$\frac{\rho(X^{(1)},t)}{\rho_0(X^{1}) = \rho(X^{(1)},0)} = \det F^{(1)} \qquad F^{(1)} = \nabla f^{(1)}_{/x^{(1)}}$$

the same ?

$$\frac{\rho(X^{(2)},t)}{\rho_0(X^{(2)}) = \rho(X^{(2)},0)} = \det F^{(2)}$$

divide the two relations to get

$$\frac{\rho_0(X^{(2)})}{\rho_0(X^{(1)})} = \frac{\det F^{(1)}}{\det F^{(2)}}$$

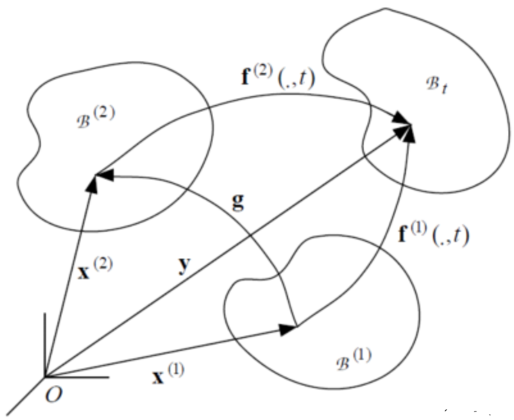
$$x^{(2)} = g(X^{(1)}) \Rightarrow F^{(1)}_{ij} = \frac{\partial f_i}{\partial X^{(1)}_j} = \frac{\partial f_i}{\partial x^{(2)}_k}\, \underbrace{\frac{\partial X^{(2)}_k}{\partial X^{(1)}_j}}_{\nabla g} = F^{(2)}_{ik}\, \nabla g_{kj}$$

$$\frac{J_t(X^{(1)})}{\rho_0(X^{(1)})} = \frac{\det F^{(1)}}{\det F^{(2)}}$$

$$\det F^{(1)} = \det F^{(2)} \det \nabla g$$

$\Longrightarrow$ $\boxed{F^{(1)} = F^{(2)} \nabla g}$ we showed this before as well

$$\boxed{\frac{\rho_0(X^{(2)})}{\rho_0(X^{(1)})} = \det \nabla g}$$



(I) $$\boxed{\underbrace{\rho_0(X^{(2)})}_{\text{"new density"}} = \underbrace{\left(\det \nabla g\right)}_{J} \underbrace{\rho_0(X^{(1)})}_{\text{"old density"}}}$$

Similar $\rho(X,t) = J\, \rho(X, 0)$

As the minimum requirement going through different initial configurations, we want the initial density to not change.

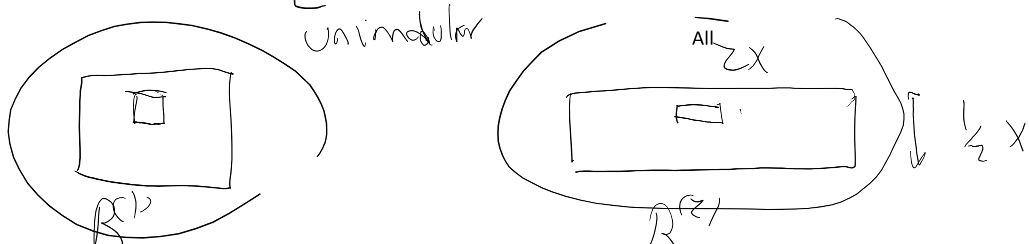
From (I) we require $\boxed{\det \nabla g = 1}$

A 2nd order tensor H for which $\det H = 1$ is called unimodular

Examples:

(1) $\nabla g = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$ $\det \nabla g = 1$ ρ_0 does not change :)

unimodular



(2) $\nabla g = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ $\det \nabla g = 4 \neq 1$

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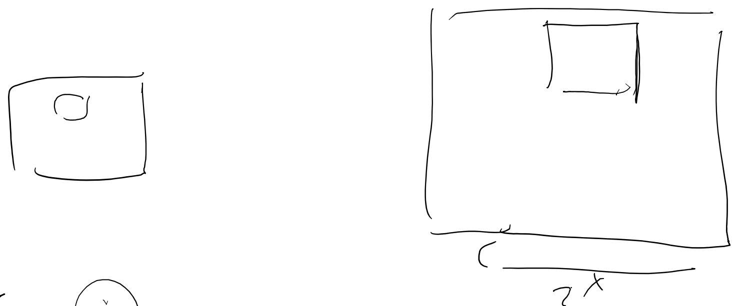
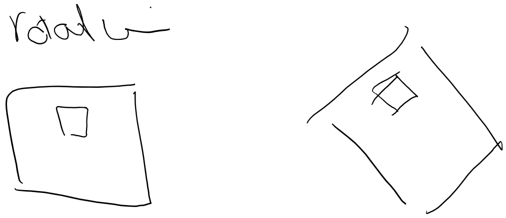


this transformation from $\chi^{(1)}$ to $\chi^{(2)}$ does not preserve density

(3) $\nabla g = Q$ rotation



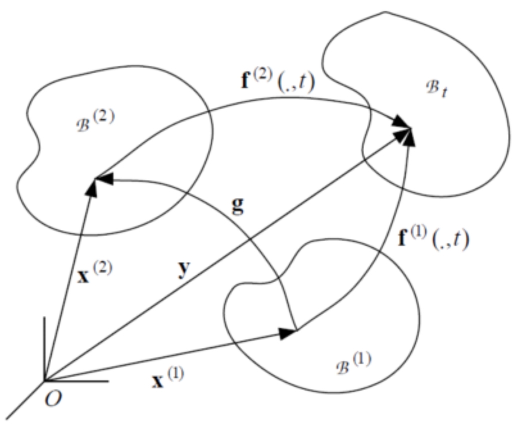
density does not change

all rotations are unimodular

Basically, if we only restric ourselves to g's for which initial density does not change (very reasonable restriction), we are left with grad g's in the space of unimodular tensors (which obviously include rotations):

$$\nabla g \in \mathrm{Unimod}^{+} = \{ H \in \underbrace{\mathrm{Lin}^{+}}_{\det H > 0} \mid \det H = 1 \}$$

How about constitutive equation for Cauchy stress tensor T (γ)



$$T = G^{(1)}(F^{(1)}, X^{(1)})$$

$$T = G^{(2)}(F^{(2)}, X^{(2)})$$

$$F^{(1)} = \nabla y_{/X^{(1)}} = \nabla y_{/X^{(2)}} \nabla X^{(2)}_{/X^{(1)}} = F^{(2)} \nabla g$$

$$\boxed{T = G^{(1)}(\underbrace{F^{(2)} \nabla g}_{F^{(1)}}, X^{(1)}) = G^{(2)}(F^{(2)}, X^{(2)})}$$

we had this relation before, basically const. eqn G^{1} is given

$G^{(2)}$ can be derived

$$\underbrace{G^{(2)}(F, X^{(2)}) = G^{(1)}(F \nabla g, X^{(1)})}_{\text{always correct}}$$

if ∇g is in symmetry group of constitutive equation what should we have?

$$\begin{cases} G^{(1)}(F, X^{(1)}) \\ G^{(2)}(F, X^{(2)}) \end{cases}$$ these should be equal

$$G^{(2)}(F, X^{(2)}) = G^{(1)}(F\nabla g, X^{(1)})$$ Always true

$$G^{2}(F, X^{(2)}) = G^{(1)}(F, X^{(1)})$$ true if ∇g belongs to symmetry group for const. Eqn (from $B_1^{(1)}$ & $B^{(2)}$ same const eqn is charaterized)

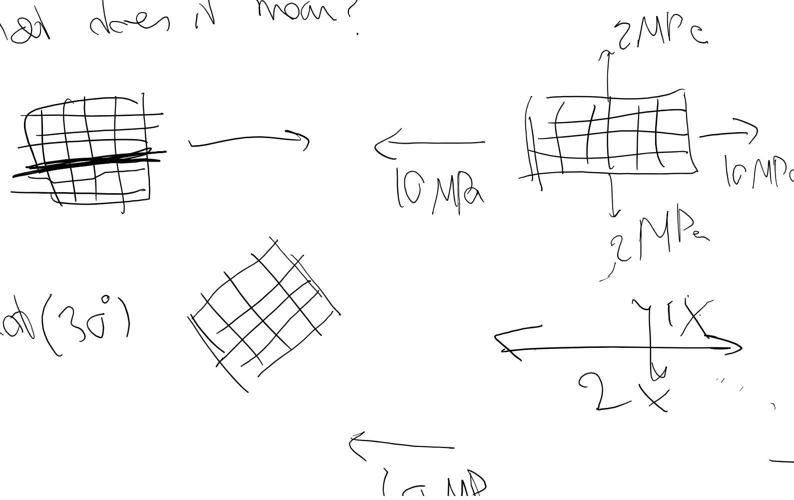
$$\boxed{G^{(1)}(F, X^{(1)}) = G^{(1)}(F\nabla g, X^{(1)})}$$

Basically, map g belongs to symmetry group for cons. Eqn. If

$$\boxed{G(F\nabla g, X) = G(F, X)}$$

we require $\det \nabla g = 1$ (to not change density by map g)

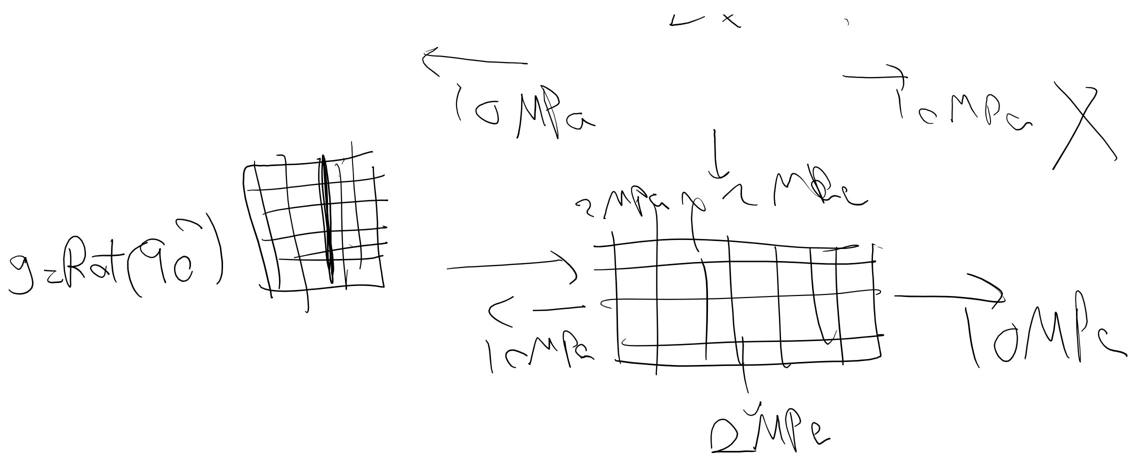
What does it mean?



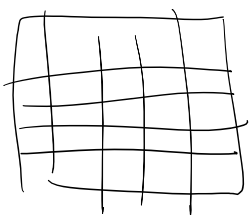
$$F = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$g = \mathrm{Rot}(30°)$

$$F\nabla g = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \mathrm{Rot}(30°)$$



stress tensor does not change



$$\mathrm{Sym}_G = \{ H \in \mathrm{Unim} \mid H = \mathrm{Rot}(k\,90^\circ) \}$$

Definition 112 (Noll, 1958) *Given an elastic body and a reference configuration that corresponds to the region* $\overset{0}{\mathcal{B}}$, *the* material symmetry group *at the material point identified by* $\mathbf{x}$ *in the reference configuration is the set*

$$\underset{\mathbf{x}}{\mathrm{Msg}} = \{\mathbf{H} \in \mathrm{Unim}\ \mathcal{V}^+ : \mathbf{G}(\mathbf{FH}, \mathbf{x}) = \mathbf{G}(\mathbf{F}, \mathbf{x})\ \forall\ \mathbf{F} \in \mathrm{Lin}\ \mathcal{V}^+\}.$$

∇g

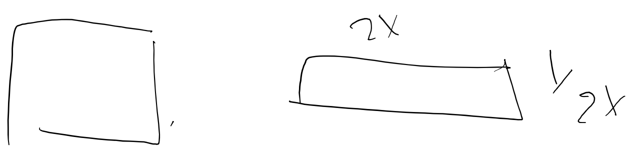
Again, it should be emphasized that the material symmetry group is characterized by tensors $\mathbf{H}$ that correspond to the gradients at $\mathbf{x}$ of deformations — not the deformations themselves. This is because the mass density and the elastic response function in the second reference configuration depend only on the gradient of the connecting deformation. Also, note that $\mathbf{H} \in \mathrm{Msg}_{\mathbf{x}}$ is not a tensor field, but rather the value of a tensor field at $\mathbf{x}$.

The following theorem presents a property of all orthogonal elements of $\mathrm{Msg}_{\mathbf{x}}$ that derives from the Principle of Material Frame-Indifference.

For isotropic solid material, what should be symmetry group of constitutive equation?

All rotations belong to symmetry group.

What is the material that ALL unimodal tensors belong to its symmetry group?



"Elastic fluids" would have all unimodal tensors in their const. eqn. symmetry group:

inviscid

$$G(FH) = G(F) \quad \forall H \ni \det H = 1$$

one can show because of

$G(F)$ reduces to simple relation

$$G(F) = \bar{G}(\det F) = \bar{G}\left(\frac{\rho}{\rho_0}\right) = \bar{\bar{G}}(\rho)$$

J

$$T = -pI = \bar{\bar{G}}(\rho, \theta)$$

temperature, ...

Theorem 174:

Let $Q \in \mathrm{Orth}\,V^+$ rotation then $Q \in M_{sg_\chi}$ iff

symmetry group for const. eqn

iff $$\boxed{\forall F \in \mathrm{Lin}\,V^+ \quad G(QFQ^t, x) = Q\,G(F, x)\,Q^t}$$

$Q \in M_{sg_\chi}$

① $G(FQ) = G(F)$ RSide

② $G(QF) = Q\,G(F)\,Q^T$ LSide

(if $Q \notin M_{sg}$ this eqn doesn't hold)

objectivity Always true

$F' = QFQ^t \xrightarrow{①} G(F'Q) = G(F') \rightarrow G((QFQ^t)Q) = G(QFQ^t)$

$\rightarrow G(QF) = G(QFQ^t)$

objectivity $\quad Q\,G(F)\,Q^t = G(QFQ^t)_{\mathrm{proof}}$

$$\boxed{G(QFQ^t, x) = QG(F, x)Q^t}$$

TAM551: using this equation, it can be shown that for linear isotropic solid, there are only two independent material parameters (E, nu OR Lame's parameters)

$E, \nu \qquad \lambda, \mu$

$$\sigma_{ij} = C_{ijkl}\,\epsilon_{kl}$$

$$C_{ijkl} = \lambda\,\delta_{ij}\,\delta_{kl} + \mu\left(\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk}\right)$$

$$\boxed{C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \underbrace{\mu}_{\text{shear modulus}} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ 2\epsilon_{12} \end{bmatrix}$$

2D