2019/11/18

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Hyperelastic materials:
It's a more restricted group of elastic materials where internal energy density is written as a function of deformation gradient:

General elastic $\quad T = G(F, x)$

$= F\,\bar{G}(C, x)F^T$ using objectivity

hyperelastic $\quad \varepsilon(F) \longrightarrow \bar{\varepsilon}(C)$

Internal energy density $\quad$ objectivity $\quad$ derive the form of const. eqn

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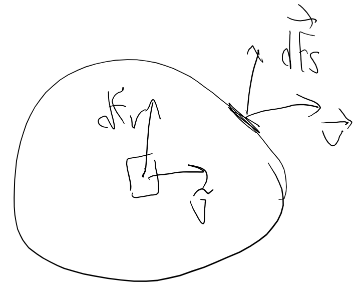
Theorem 158 (Theorem of Expended Power) *If the linear momentum is balanced, then* $\forall$ *parts* $\mathcal{P} \overset{0}{\subset} \mathcal{B}$ *and* $\forall$ *times* $t \in [t_0, \infty)$,

$$\frac{d}{dt}\int_{\mathcal{P}_t} \frac{1}{2}\left|\hat{\mathbf{v}}(\mathbf{y},t)\right|^2 \rho(\mathbf{y},t)\,dV_y = \int_{\partial\mathcal{P}_t} \mathbf{t}_{\mathbf{n}(\mathbf{y},t)}(\mathbf{y},t)\cdot\hat{\mathbf{v}}(\mathbf{y},t)\,dA_y + \int_{\mathcal{P}_t} \mathbf{b}(\mathbf{y},t)\cdot\hat{\mathbf{v}}(\mathbf{y},t)\rho(\mathbf{y},t)\,dV_y - \int_{\mathcal{P}_t} \mathbf{T}(\mathbf{y},t)\cdot\mathbf{L}(\mathbf{y},t)\,dV_y.$$



$$\underbrace{\frac{D}{Dt}\int \frac{1}{2}|v|^2 \underbrace{\rho\, dV_y}_{dm}}_{\text{kinetic energy}} = \underbrace{\int (\underbrace{t_n}_{\text{traction}}\, dA_y)\cdot \hat{v}}_{\substack{dF_s \\ \text{power from surface forces}}} + \underbrace{\int \rho \; \hat{v}}_{\substack{\vec{dF_v} \\ \text{power from body force}}} - \underbrace{\int T . L\, dV_y}_{\text{power from internal elastic deformat}}$$

$$L = \nabla_y \hat{v} = \operatorname{grad} \hat{v} \qquad L_{ij} = \frac{\partial v_i(y,t)}{\partial y_j}$$

Spatial gradient velocity

$D = \operatorname{sym} L = \frac{1}{2}(L + L^T)$ stretching tensor

$W = \operatorname{asym} L = \frac{1}{2}(L - L^T)$, Spin tensor

Don't confuse D with stretch tensor U

$$U = \sqrt{C} = \sqrt{F^t F}$$

Theorem 158 (Theorem of Expended Power) *If the linear momentum is balanced, then $\forall$ parts $\mathcal{P} \subset \overset{0}{\mathcal{B}}$ and $\forall$ times $t \in [t_0, \infty)$,*

$$\frac{d}{dt}\int_{\mathcal{P}_t} \frac{1}{2}|\hat{\mathbf{v}}(\mathbf{y},t)|^2 \rho(\mathbf{y},t)\, dV_y = \int_{\partial\mathcal{P}_t} \underline{\mathbf{t}_{\mathbf{n}(\mathbf{y},t)}(\mathbf{y},t)\cdot\hat{\mathbf{v}}(\mathbf{y},t)}\, dA_y + \int_{\mathcal{P}_t} \mathbf{b}(\mathbf{y},t)\cdot\underline{\hat{\mathbf{v}}(\mathbf{y},t)}\rho(\mathbf{y},t)\, dV_y - \int_{\mathcal{P}_t} \mathbf{T}(\mathbf{y},t)\cdot\mathbf{L}(\mathbf{y},t)\, dV_y.$$

Balance on Lin momentum

$$\underbrace{\frac{D}{Dt}\int_{P_t} \rho v\, dV_y = \int_{\partial P_t} t_n\, dA_y + \int_{P_t} \rho b\, dV_y}$$

get balance of linear momentum $\times\ \hat{v}$

Modify terms

$$\frac{D}{Dt}\int_{P_t} v\,\underbrace{(\rho\, dV_y)}_{dm} = \int_{P_t} \frac{Dv}{Dt}\,\underbrace{\rho\, dV_y}_{dm}$$

reduced transport theorem

$$\int_{\partial P_t} t_n\, dA_y = \int_{\partial P_t} \underbrace{T \cdot n\, dA_y}_{\sigma} = \int_{P_t} \operatorname{div} T\, dV_y$$

Use localization to get

(1) $$\boxed{\rho \frac{D}{Dt} v = \operatorname{div} T + \rho b}$$

as we had seen this before, it's equation of motion / EOM / strong form of balance of lin. momentum

Inner product with velocity:

$$\boxed{\hat{v}\cdot\rho\frac{Dv}{Dt} = \frac{\rho}{2}\frac{D|v|^2}{Dt}} \quad (3)$$

(2) $$\boxed{\hat{v}\cdot\Big(\underbrace{\rho\frac{D}{Dt}v}_{(3)} - \underbrace{\operatorname{div} T}_{(4)} - \rho b\Big) = 0}$$

$$\boxed{\hat{v}\cdot\operatorname{div} T = \operatorname{div}(Tv) - L : T} \quad (4)$$

Let's look at terms in (2)

$$\hat{v}\cdot\rho\frac{D\hat{v}}{Dt} = \rho\frac{D}{Dt}\frac{1}{2}|v|^2$$

we claim this.

$$\frac{D}{Dt}\frac{1}{2}|v|^2 = \frac{D}{Dt}\frac{1}{2}\hat{v}\cdot\hat{v} = \frac{D}{Dt}\frac{1}{2}v_i \cdot v_i = \frac{1}{2}\left(\frac{Dv_i}{Dt}v_i + v_i\frac{Dv_i}{Dt}\right) = v\,\frac{Dv_i}{Dt}$$

$$\hat{v}\cdot\operatorname{div} T = v_i T_{ij,j} = \underbrace{(v_i T_{ij})_{,j} - v_{i,j} T_{ij}} = (T^t_{ji} v_i)_{,j} - L_{ij} T_{ij}$$

$$= \operatorname{div}(T^t v) - L : T \qquad (A : B = A_{ij} B_{ij}) \qquad L_{ij} = \frac{\partial v_i}{\partial y_j}$$

$$= \operatorname{div}(Tv) - L : T \qquad T = T^t$$

Plug 3 and 4 in equation 2 to get:

$$\rho \frac{D}{Dt} \frac{1}{2}|v|^2 - \operatorname{div}(Tv) + L:T - \hat{v}\cdot\rho b = 0$$

integrate this over P_t to get

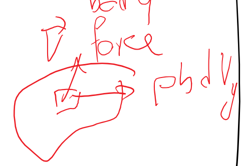
$$\underbrace{\int_{P_t} \frac{D}{Dt}\frac{1}{2}|v^2|\underbrace{\rho\, dV_y}_{dm}}_{\text{reduced transport}} - \int_{P_t} \operatorname{div}(Tv)\,dV_y + \int_{P_t} L:T\,dV_y - \int_{P_t} \hat{v}\rho b\,dV_y = 0$$

$$\frac{D}{Dt}\int \frac{\rho}{2}|v^2|\,dV_y - \int_{\partial P_t} \underbrace{Tv\cdot n\,dA_y}_{\text{Divergence theorem}} + \int_{P_t} L:T\,dV_y - \int_{P_t}\hat{v}\rho b$$

$v\cdot T^t n$ " Tn

theorem of expended power — is equal to $(Tn)\cdot v$

$$\underbrace{\frac{D}{Dt}\int_{P_t}\frac{1}{2}|v^2|\rho\,dV_y}_{\text{Rate of kinetic energy}} + \int_{P_t} L:T\,dV_y = \underbrace{\int_{\partial P_t} \vec{v}\cdot\underset{Tn}{\vec{t}}\,dA_y}_{\text{Power from forces on } \partial P_t} + \underbrace{\int_{P_t}\hat{v}\rho b\,dV_y}_{\text{Power from body force}}$$



Balance of energy

$$\frac{D}{Dt}\int\left(\frac{\rho}{2}|v|^2 + \rho e\right)dV_y = \int \vec{v}\cdot\vec{t}\,dA_y + \int\hat{v}\cdot\rho b\,dV_y + (\text{other terms})$$

e: internal energy density

other terms: EM, thermal, ... not considered here

(4)

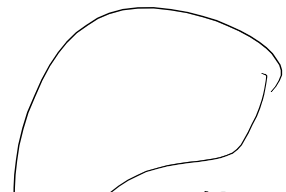
Comparison of 2 eqns

$$\boxed{\frac{D}{Dt}\int_{P_t} \underbrace{\rho}\, \underbrace{e\, dV_y} = \int_{P_t} L : T\, dV_y} \qquad (5)$$

reduced transport

$$\int_{P_t} \rho \frac{De}{Dt}\, dV_y = \int_{P_t} L : T\, dV_y$$

$$\rightarrow \forall P_t \qquad \boxed{\int_{P_t}\left(\rho\frac{De}{Dt} - L:T\right)dV_y = 0}$$



Since P_t is arbitrary use localization to get

(★) $$\boxed{\rho \frac{De}{Dt} = T : L \qquad L = \nabla_y \hat{v}}$$

Hyperelastic material:

Internal energy density e can be written as a function of deformation gradient for hyperelastic materials (not all materials are hyperelastic):

hyperelastic

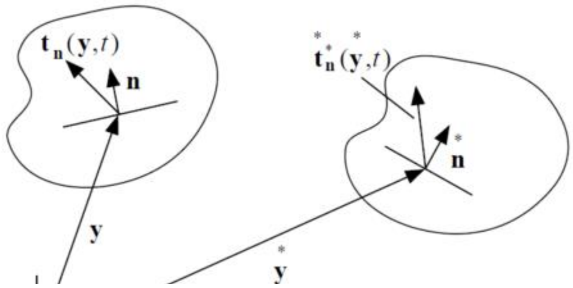
$$e(F) \xrightarrow{\text{use (★)}} T = F\bar{G}(C)F = G(F)$$

elastic

elastic

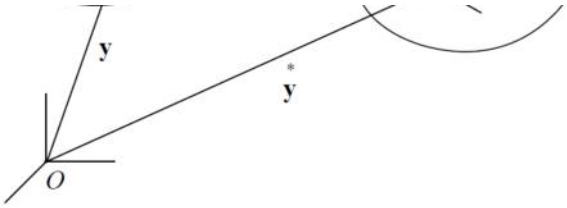
to get G for hyperelastic

First thing we use objectivity:



$$\overset{*}{y} = c(t) + Q(t)\, y(x,t)$$

$$\left.\begin{aligned} n^* &= Qn \\ t^* &= Qt \end{aligned}\right\} \text{ vectors}$$



$t^* = Qt$ /

$T^* = QTQ^t$ 2nd order tensor

$e^* = e$
temperature $\theta^* = \theta$ } scalars

Use objectivity to restrict $e(F)$ (hyperelastic)

$e(F^*) = e(F)$

$\downarrow$

$e(QF) = e(F)$

$$F^*_{ij} = \frac{\partial y^*_i}{\partial x_j} = \frac{\partial \left(c_i(t) + Q_{im}\, y_m(x,t)\right)}{\partial x_j} = Q_{im}\frac{\partial y_m}{\partial x_j} \Rightarrow \boxed{F^* = QF}$$

$F = RU$

$\boxed{e(QRU) = e(F)}$ choose $Q = R^t$ to get

$e(\sqrt{C}) = e(U) = e(F)$ we can write e as a function of U rather than F

Theorem 179 *If the elastic constitutive equation for the internal energy*

e energy density

$$\tilde{\varepsilon}(\mathbf{x}, t) = \bar{\varepsilon}(\mathbf{F}(\mathbf{x}, t), \mathbf{x})$$

satisfies the Principle of Material Frame-Indifference, then it can be written in either of the following reduced forms:

$$\tilde{\varepsilon}(\mathbf{x}, t) = \bar{\varepsilon}(\mathbf{U}(\mathbf{x}, t), \mathbf{x}); \tag{4.7}$$
$$\tilde{\varepsilon}(\mathbf{x}, t) = \bar{\varepsilon}(\mathbf{C}(\mathbf{x}, t), \mathbf{x}). \tag{4.8}$$

Conversely, each of these reduced constitutive equations automatically satisfies the scalar Principle of Material Frame-Indifference $\forall\, \hat{\varepsilon}$ *or* $\bar{\varepsilon}$.

★★

So far we have

★ $\rho \frac{De}{Dt} = T : L$
★★ $\bar{e}(C)$ objectivity

3D $C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & & C_{23} \end{bmatrix}$

$\bar{e}(C) = 2C_{12} + 2C_{21}$

3D $C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ & C_{22} & C_{23} \\ \text{sym.} & & C_{33} \end{bmatrix}$ $\quad C = F^t F$

$\bar{e}(C) = 2C_{12} + 2C_{21}$

$$\frac{\partial \bar{e}}{\partial C} = \begin{bmatrix} \frac{\partial \bar{e}}{\partial C_{11}} & \frac{\partial \bar{e}}{\partial C_{12}} & \frac{\partial \bar{e}}{\partial C_{13}} \\ \frac{\partial \bar{e}}{\partial C_{21}} & - & - \\ & & \frac{\partial \bar{e}}{\partial C_{33}} \end{bmatrix} \qquad \left(\frac{\partial \bar{e}}{\partial C}\right)_{ij} = \frac{\partial \bar{e}}{\partial C_{ij}}$$

Def.

$\bar{e}(C) = 2C_{12} + 2C_{21}$ $\quad \frac{\partial \bar{e}}{\partial C} = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ sym

$\underline{\bar{e}(C) = C_{12} + 3C_{21}}$ $\quad \frac{\partial \bar{e}}{\partial C} = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ nonsym.

since $C_{12} = C_{21}$ $\quad \bar{e}(C)$ cases above the same

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$$\bar{C}_{12} = \frac{1}{2}(C_{12} + C_{21})$$
$$\bar{C}_{23} = \frac{1}{2}(C_{23} + C_{32})$$
$$\bar{C}_{31} = \frac{1}{2}(C_{31} + C_{13})$$

$$e(C_{11}, C_{22}, C_{33}, \bar{C}_{12}, \bar{C}_{23}, \bar{C}_{31}) \rightarrow$$ 6 independ components of C

$$\frac{\partial e}{\partial C_{12}} = \frac{\partial e}{\partial \bar{C}_{12}} \frac{\partial \bar{C}_{12}}{\partial C_{12}} = \frac{1}{2} \frac{\partial e}{\partial \bar{C}_{12}}$$

$$\frac{\partial e}{\partial C_{21}} = \frac{\partial e}{\partial \bar{C}_{12}} \underbrace{\frac{\partial \bar{C}_{12}}{\partial C_{21}}}_{1/2} = \frac{1}{2} \frac{\partial e}{\partial \bar{C}_{12}}$$

This is explained in more detail:

$$
\begin{aligned}
\bar{C}_1 &= C_{11},\ \bar{C}_2 = C_{22},\ \bar{C}_3 = C_{33},\\
\bar{C}_4 &= \frac{C_{12}+C_{21}}{2},\ \bar{C}_5 = \frac{C_{23}+C_{32}}{2},\ \bar{C}_6 = \frac{C_{13}+C_{31}}{2},
\end{aligned}
$$

and write

$$
\begin{aligned}
\bar{\varepsilon}(\mathbf{C}) &= \bar{\varepsilon}_S(\bar{C}_1,\bar{C}_2,\bar{C}_3,\bar{C}_4,\bar{C}_5,\bar{C}_6)\\
&=: \bar{\varepsilon}_L(C_{11},C_{22},C_{33},C_{12},C_{21},C_{23},C_{32},C_{13},C_{31}).
\end{aligned}
$$

By the Chain Rule we have

$$\frac{\partial\bar{\varepsilon}_L}{\partial C_{ij}} = \sum_{\Gamma=1}^{6}\frac{\partial\bar{\varepsilon}_S}{\partial\bar{C}_\Gamma}\frac{\partial\bar{C}_\Gamma}{\partial C_{ij}},$$

so, in particular,

$$\frac{\partial\bar{\varepsilon}_L}{\partial C_{12}} = \frac{\partial\bar{\varepsilon}_S}{\partial\bar{C}_4}\frac{\partial\bar{C}_4}{\partial C_{12}} = \frac{1}{2}\frac{\partial\bar{\varepsilon}_S}{\partial\bar{C}_4}.$$

and

$$\frac{\partial\bar{\varepsilon}_L}{\partial C_{21}} = \frac{1}{2}\frac{\partial\bar{\varepsilon}_S}{\partial\bar{C}_4}.$$

In general, we have

$$\frac{\partial\bar{\varepsilon}_L}{\partial C_{pq}} = \frac{\partial\bar{\varepsilon}_L}{\partial C_{qp}}.$$

$\star$ $\rho\frac{De}{Dt} = T : L$

$\star\star$ $e = \hat{e}(C)$

$$\frac{De(C)}{Dt} = \underbrace{\frac{\partial e'}{\partial C_{ij}}}_{\text{sym.}}\frac{DC_{ij}}{Dt}$$

sym. tensor we just talked about

$$\boxed{\frac{De}{Dt} = \left(\frac{\partial e}{\partial C}\right)\frac{DC}{Dt}}$$

$$C = F^tF \qquad \frac{DC}{Dt} = \underbrace{\left(\frac{DF}{Dt}\right)^t}_{\dot{F}^t}F + F^t\underbrace{\frac{DF}{Dt}}_{\dot{F}}$$

$$\dot{F}_{im} = \frac{D}{Dt}F_{im} = \frac{D}{Dt}\left(\frac{\partial y_i}{\partial x_m}\right)\Big|_{x\text{-fixed}} = \frac{\partial^2 y_i}{\partial t\,\partial x_m}\Big|_{x\text{-fixed}}$$

$$\frac{\partial}{\partial x_m}\underbrace{\left(\frac{\partial y_i}{\partial t}\right)}\Big|_{x\text{-fixed}} = \frac{\partial}{\partial x_m}v_i = \underbrace{\frac{\partial v_i}{\partial y_m}}_{L_{im}}\underbrace{\frac{\partial y_m}{\partial x_j}}_{F_{mj}}$$

$$\boxed{\dot{F} = LF,\quad \dot{C} = \dot{(F^tF)} = \dot{F}^tF + F^t\dot{F}} \rightarrow$$

$$\dot{C} = (F^tL^t)F + F^t(LF) \Rightarrow$$

$$C = (F^t L^t)F + F^t(LF) \Longrightarrow$$

$$\dot{C} = 2F^t\underbrace{\left(\frac{L+L^t}{2}\right)}_{\text{stretching tensor } D}F$$

$$\boxed{\dot{C} = 2F^tDF}$$

$$\rho\frac{De}{Dt} = \rho\left(\frac{\partial e}{\partial C}\right)\underbrace{\frac{DC}{Dt}}_{\dot{C}} = \rho\left(\frac{\partial e}{\partial C}\right)(2F^tDF)$$

$$= T:L$$

can show $\quad 2\rho\frac{\partial e}{\partial C}F^tDF = 2\rho\left(F\frac{\partial e}{\partial C}F^t\right):D$

$$\boxed{\underbrace{2\rho\left(F\frac{\partial \bar{e}}{\partial C}F^t\right)}_{S}:D = T:L}$$

$L = \nabla v$

$D = \operatorname{sym} L$

symmetric

$T = ?$

$$S^t = 2\rho(F^t)^t\underbrace{\left(\frac{\partial \bar{e}}{\partial C}\right)^t}_{\text{sym}}F$$

$$\underset{\text{sym}}{S}:(L) = S:(D+W) = S:\underset{\substack{\text{sym}\\ \text{stretching tensor}}}{D} + \cancel{\underset{\text{sym}}{S}:\underset{\substack{\text{skew sym}\\ \text{spin}}}{W}}$$

$$2\rho F^t\frac{\partial\bar{e}}{\partial C}F\ldots \quad T$$

tensor

$$\boxed{\underbrace{2\rho F^t \frac{\partial \bar{e}}{\partial C} F}_{S} : L = T : L} \quad \forall L$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow S_{11} = T_{11}$$

hyperelastic

$$T = 2\rho F^t \frac{\partial \bar{e}}{\partial C} F$$

hyperelastic

$$= F^t \underbrace{\bar{G}(C)}\, F$$

elastic considering objectivity