2019/11/20

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Last time we derived constitutive equation for hyperelastic materials

$T = G(F) = \underline{F\bar{G}(C)F^t}$ using objectivity

elastic

Hyperelastic

$\underline{e = e(F)} = \bar{e}(C)$ using objectivity

$$\Rightarrow \boxed{\begin{gathered} T = 2\rho F \frac{\partial \bar{e}(C)}{\partial C} F^t = F \underbrace{\left(\frac{2\rho_0}{\sqrt{\det C}} \frac{\partial \bar{e}(C)}{\partial C} \right)}_{\bar{G}(C)} F^t \\ \rho = \frac{\rho_0}{J} = \frac{\rho_0}{\det F} = \frac{\rho_0}{\sqrt{\det C}} \\ \text{Hyperelastic} \quad \bar{G}(C) = \frac{2\rho_0}{\sqrt{\det C}} \frac{\partial \bar{e}(C)}{\partial C} \end{gathered}}$$

Linearizing constitutive equation:
For infinitesimal theory

$\longrightarrow S = \mathbb{C} E$

E → linear (infinitesimal) strain

S → PK-I

$\mathbb{C}$ → 4th order elasticity tensor

$$S_{ij} = \mathbb{C}_{ijkl} E_{kl}$$

$\mathbb{C}_{ijkl}$ → 4th order elasticity tensor

EOM $\quad \operatorname{div} T + \rho b = \rho a \quad$ Eulerian

$\quad \underline{\operatorname{Div} S} + \rho_0 b = \rho_0 a \quad$ Lagrangian

→ want to find constitutive eqn for S in infinitesimal regime

$$\left.\begin{aligned} S &= J\, T F^{-t} \\ T &= \underbrace{F \bar{G}(C) F^t}_{\text{general elastic material}} \end{aligned}\right\} \longrightarrow \text{approximate form of } S \text{ in terms of } E$$

T → Cauchy stress (σ)

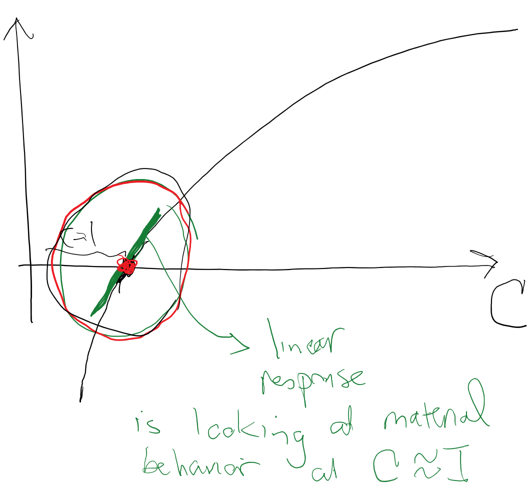
Background $\quad f(x_0 + \Delta x) = f(x_0) + \Delta x f'(x_0) + \underbrace{\frac{\Delta x^2}{2} f''(x_0) + \dots}_{\overset{R(\Delta x)}{O(\Delta x^2)}, \ \underline{o(\Delta x)}}$

$$R(\Delta x) \leq K (\Delta x)^2$$

$\frac{R(\Delta x)}{\Delta x} \longrightarrow 0$ as $\Delta x \to 0$

the remander $R(\Delta x)$ goes to zero faster than Δx

Derivation of linear elasticity $P \approx I \longleftarrow S$



1D $F = \frac{dy}{dx} = \frac{du}{dx} + 1$ ND

$C = F^t F = (u' + 1)^2$

Most often (but by no means a general statement), S = 0 at C = Identity

$$S = J T F^{-t}$$
$$T = F \bar{G}(C) F^t$$
$$\longrightarrow \boxed{S = J F \bar{G}(C)}$$

We will expand G(C) around C = I:

$\bar{G}(C)$ expanding this around I:

$\bar{G}_{ij}(C) = \bar{G}_{ij}((C - I) + I)$ do a Taylor's expansion around I

$$= \bar{G}_{ij}(I) + \frac{\partial \bar{G}_{ij}(I)}{\partial C_{kl}}(C-I)_{kl} + o(C-I)$$

like

$$f(x_0 + \Delta x) = f(x_0) + f'(x_0)\underbrace{\Delta x}_{C-I} + o(\Delta x)$$

$$\boxed{C_{ijkl} = 2 \left. \frac{\partial \bar{G}_{ij}(C)}{\partial C_{kl}} \right|_{\text{at } C=I}}$$

$\bar{G}_{ij}(C) = \bar{G}_{ij}(I) + \ldots$ 0

$$\bar{G}_{ij}(C) = \bar{G}_{ij}(I) +$$

$$\frac{1}{2} C^0_{ijkl}(C-I)_{kl} + o(C-I)$$

$$(1) \quad \boxed{\bar{G}_{ij}(C) = \frac{1}{2} C^0_{ijkl}(C-I)_{kl} + o(C-I)}$$

$$C - I = (F^tF - I) = (H^t + I)(H+I) - I = H^t + H + H^tH$$

$$= 2\underbrace{\left(\frac{H^t+H}{2}\right)}_{E} + O(\varepsilon^2) \qquad \varepsilon = \text{norm of } H$$

$$(2) \quad \boxed{C - I = 2E + O(\varepsilon^2)}$$

plug (2) in (1) to get

$$C - I = 2E + H^tH = O(\varepsilon)$$

$$\bar{G}_{ij}(C) = \frac{1}{2} C^0_{ijkl}(2E_{kl} + O(\varepsilon^2)) + o(\overbrace{O(\varepsilon)}^{C-I})$$

$$(3) \quad \boxed{\bar{G}_{ij}(C) = C^0_{ijkl}E_{kl} + o(\varepsilon)}$$

$$S = JF\bar{G}(C)$$

plug (3) in

$$S_{mj} = J F_{mi} \bar{G}_{ij}(C)$$

$$= J(\delta_{mi} + \underbrace{H_{mi}}_{O(\varepsilon)})(C^0_{ijkl}E_{kl} + o(\varepsilon))$$

$$= (I + \underbrace{\text{trace}(E) + o(\varepsilon^2)}_{O(\varepsilon)})(\delta_{mi} + O(\varepsilon))(C_{ijkl}E_{kl} + o(\varepsilon))$$

bigger term

$$S_{mj} = 1 \cdot \delta_{mi} C^0_{ijkl} E_{kl} + o(\varepsilon)$$

$$S_{mj} = C^0_{mjkl} E_{kl} + o(\varepsilon)$$

$$\boxed{S_{ij} = C^0_{ijkl} E_{kl} + o(\varepsilon)}$$

$$S_{ij} = \mathbb{C}_{ijkl} E_{kl} + o(\varepsilon)$$

$$S = \mathbb{C} E + o(\varepsilon)$$

$$\mathbb{C} = 2 \frac{\partial \bar{G}(C)}{\partial C} (C = I)$$

In practice we use

Is this objective? $\leftarrow$ $S = \mathbb{C} E$

S = CE is not objective but the error in violating it is very small
(o(epsilon))

$S = JF\bar{G}(C)$ is objective

PK-II $\leftarrow$ $S = \mathbb{C} E$ $\quad S_{ij} = \mathbb{C}_{ijkl} E_{kl}$

$\downarrow$ P $\rightarrow$ PK-I

T $\rightarrow$ Cauchy

We can use any of the stress tensors S, P, T and the same relation holds.

How many terms do we have in 4th order elasticity tensor

$\mathbb{C}_{ijkl}$ $\quad i,j,k,l \in \{1,2,3\}$

$3^4 = 81$

independent

$$S = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \qquad E = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix}$$

Sym within o(ε) infinitesimal theory $\qquad$ Sym.

only $6 \times 6 = 36$ terms of $\mathbb{C}$ are independent using the sym. of E and $S = T$ (infinitesimal theory)

Symmetries of the 4th order elasticity tensor:

1 — Minor symmetries

$$C_{ijkl} = C_{jikl} \quad \checkmark$$

Always true

$$C_{ijkl} = C_{ijlk} \quad \checkmark$$

81 independent terms -> only 36 independent terms

2. Major symmetry

$$C_{ijkl} = C_{klij}$$

Can be proven only for HYPERELASTIC materials
36 independent components of C ->
21 independent components for hyperelastic material

Proofs:

(A) $C_{ijkl} = C_{jikl}$

$$T = T^t \quad \begin{cases} T = F\,\bar{G}(C)\,F^t \\ T^t = F\,\bar{G}^t(C)\,F^t \end{cases} \rightarrow \quad F\bar{G}(C)F^t = F\bar{G}^t(C)F^t$$

Balance of angular momentum

$$\boxed{\bar{G}(C) = \bar{G}^t(C) \longrightarrow \bar{G} \text{ is symmetric}}$$

$$C_{ijkl} = 2\,\frac{\partial \bar{G}_{ij}(C)}{\partial C_{kl}}(C = I) = \frac{\partial \bar{G}_{ji}(C)}{\partial C_{kl}}(C = I) \quad \bar{G} = \bar{G}^t$$

$$= C_{jikl}$$

2nd minor symmetry:

$$C_{ijkl} = 2\,\frac{\partial \bar{G}_{ij}(C)}{\partial C_{kl}}(C = I) = 2\,\frac{\partial \bar{G}_{ij}(C)}{\partial C_{lk}}(C = I)$$

Bottom line this is because of symmetry of C where Ckl = Clk and need to treat only one as independent parameter

This is similar to argument from last time:

$$\bar{C}_{12} = \frac{1}{2}(C_{12} + C_{21})$$
$$\bar{C}_{23} = \frac{1}{2}(C_{23} + C_{32})$$
$$\bar{C}_{31} = \frac{1}{2}(C_{31} + C_{13})$$

$$e(\boxed{C_{11}, C_{22}, C_{33}, \bar{C}_{12}, \bar{C}_{23}, \bar{C}_{31}}) \rightarrow \text{6 independent components of } C$$

$$\frac{\partial e}{\partial C_{12}} = \frac{\partial e}{\partial \bar{C}_{12}} \frac{\partial \bar{C}_{12}}{\partial C_{12}} = \frac{1}{2} \frac{\partial e}{\partial \bar{C}_{12}}$$

$$\frac{\partial e}{\partial C_{21}} = \frac{\partial e}{\partial \bar{C}_{12}} \underbrace{\frac{\partial \bar{C}_{12}}{\partial C_{21}}}_{1/2} = \frac{1}{2} \frac{\partial e}{\partial \bar{C}_{12}}$$

This is explained in more detail:

$$\bar{C}_1 = C_{11},\ \bar{C}_2 = C_{22},\ \bar{C}_3 = C_{33},$$
$$\bar{C}_4 = \frac{C_{12} + C_{21}}{2},\ \bar{C}_5 = \frac{C_{23} + C_{32}}{2},\ \bar{C}_6 = \frac{C_{13} + C_{31}}{2},$$

and write

$$\bar{\varepsilon}(\mathbf{C}) = \bar{\varepsilon}_S(\bar{C}_1, \bar{C}_2, \bar{C}_3, \bar{C}_4, \bar{C}_5, \bar{C}_6)$$
$$=: \bar{\varepsilon}_L(C_{11}, C_{22}, C_{33}, C_{12}, C_{21}, C_{23}, C_{32}, C_{13}, C_{31}).$$

By the Chain Rule we have

$$\frac{\partial \bar{\varepsilon}_L}{\partial C_{ij}} = \sum_{\Gamma=1}^{6} \frac{\partial \bar{\varepsilon}_S}{\partial \bar{C}_\Gamma} \frac{\partial \bar{C}_\Gamma}{\partial C_{ij}},$$

so, in particular,

$$\frac{\partial \bar{\varepsilon}_L}{\partial C_{12}} = \frac{\partial \bar{\varepsilon}_S}{\partial \bar{C}_4} \frac{\partial \bar{C}_4}{\partial C_{12}} = \frac{1}{2} \frac{\partial \bar{\varepsilon}_S}{\partial \bar{C}_4}.$$

and

$$\frac{\partial \bar{\varepsilon}_L}{\partial C_{21}} = \frac{1}{2} \frac{\partial \bar{\varepsilon}_S}{\partial \bar{C}_4}.$$

In general, we have

$$\frac{\partial \bar{\varepsilon}_L}{\partial C_{pq}} = \frac{\partial \bar{\varepsilon}_L}{\partial C_{qp}}.$$

Major symmetry (can only be proven for hyperelastic material)

For hyperelastic material we had:

$$\boxed{\bar{G}(C) = \frac{2\rho_0}{\sqrt{\det C}} \frac{\partial \bar{e}(C)}{\partial C}}$$

$$\mathcal{C} = 2 \frac{\partial \bar{G}(C)}{\partial C}(C = I)$$

$$= 4\rho\, \partial\left(\frac{2\rho_0}{} \left(\partial \bar{e}(C)\right)\right)$$

$$= 4\rho_0 \frac{\partial}{\partial C}\left(\frac{2\rho_0}{\sqrt{\det C}} \frac{\partial \bar{e}(C)}{\partial C} \right)\Big|_{C=I}$$

$$= 4\rho_0 \left(\frac{\partial \frac{1}{\sqrt{\det C}}}{\partial C}(C=I) \cancel{\frac{\partial \bar{e}(C)}{\partial C}}(C=I) \right)$$

0

$(ab)' = a'b + ab'$

$$+ \frac{4\rho_0}{\sqrt{\det C}} \frac{\partial^2 \bar{e}(C)}{\partial C \partial C}$$

$$\Rightarrow \boxed{\mathbb{C}_{ijkl} = 4\rho \frac{\partial^2 \bar{e}(C)}{\partial C_{ij} \partial C_{kl}}(C=I)}$$

$$\mathbb{C}_{ijkl} = 4\rho \frac{\partial^2 \bar{e}(C)}{\partial C_{ij} \partial C_{kl}} = 4\rho \frac{\partial^2 \bar{e}(C)}{\partial C_{kl} \partial C_{ij}} = \mathbb{C}_{klij}$$

$$\left(\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \quad \text{reason} \right)$$

For hyperelastic material we also have the major symmetry in C

$$\mathbb{C}_{ijkl} = \mathbb{C}_{klij}$$

$$S_{ij} = \mathbb{C}_{ijkl} E_{kl}$$

2nd order 4th order

$$S = T = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \end{bmatrix}$$

infinitesimal

shear stresses

6 independent

$$\sigma = T \quad \text{infinitesimal limit} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

sym

stresses
6 independent
normal stress

$$E = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix} \text{sym}$$

torsional shear strains
normal strains

$$\sigma = \begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ \hline T_{12} \\ T_{23} \\ T_{31} \end{bmatrix} \begin{matrix} \} \text{normal stress} \\ \} \text{shear stresses} \end{matrix}$$

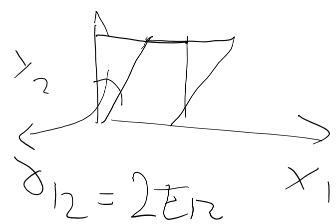
Voigt stress array

$$\gamma = \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ \hline 2E_{12} \\ 2E_{23} \\ 2E_{31} \end{bmatrix} \begin{matrix} \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{matrix}$$

→ normal strains

eng. shear strains

Voigt strain array



change of angle

$$\gamma_{12} = 2E_{12}$$

sym. only for hyperelastic

$$\sigma = \begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ \hline T_{12} \\ T_{23} \\ T_{31} \end{bmatrix} = \underbrace{\begin{bmatrix} S_{11} & S_{12} & S_{13} & \cdots & S_{16} \\ & & & & \\ S_{61} & & & & S_{66} \end{bmatrix}}_{S \text{ is your assignment}} \underbrace{\begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ \hline 2E_{12} \\ 2E_{23} \\ 2E_{31} \end{bmatrix}}_{\gamma}$$

36 components for S

→ 21 independent for hyperelastic

hyperelastic $\quad e = \frac{1}{2} T : E = \frac{1}{2} E : CE$

hyperelas

$$e = \frac{1}{2}\, T : E = \frac{1}{2}\, E : C\, E$$

$$= \frac{1}{2}\, \sigma \cdot \gamma = \frac{1}{2}\, \gamma \cdot S \gamma$$

$$= \frac{1}{2}\big(T_{11} E_{11} + T_{22} E_{22} + T_{33} E_{33} + 2 E_{12} T_{12} + 2 E_{23} T_{23}$$

$$+ T_{31} E_{31} \big)$$