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Thermodynamics (Abeyaratne Chapter 5):

First law: Balance of energy

We have already covered balance of energy:

$$\frac{D}{Dt}\left(\int_{P_t} \frac{1}{2}\rho v\cdot v\, dV_y + \int_{P_t} \rho e\, dV_y\right) = \overbrace{\int_{\partial P_t} t\cdot v\, dA_y + \int_{P_t} \rho b\cdot v\, dV_y}^{\text{mechanical}}$$

(1) (2) (3) (4)

e: specific energy density

$$\underbrace{-\int q\cdot n\, dA_y + \int Q\, dV_y}_{\text{thermal contribution}}$$

(5) $q\cdot n$: net heat flux, outward flux; (6) Q: heat source

Abeyaratne: $h \to -h$ influx, $\rho r = Q$

Expended power:

$$\frac{D}{Dt}\int_{P_t} \frac{1}{2}\rho v\cdot v\, dV_y = \int_{\partial P_t} t\cdot v\, dA_y - \int_{P_t} T:D\, dV_y + \int_{P_t} \rho b\, dV_y$$

(1) (3) (4)

subtract the 2 eqns to get

(1)

$$\boxed{\frac{D}{Dt}\int_{P_t} \rho e\, dV_y = \int_{P_t} T:D\, dV_y - \int_{\partial P_t} q\cdot n\, dA_y + \int_{P_t} Q\, dV_y}$$

for ~~dry~~ perelastic we had the same eqn w/o considering the ~~thermal~~ effects

$$\frac{D}{Dt}\int \rho e\, dV_y = \int_{P_t} \rho \frac{De}{Dt}\, dV_y \qquad \int_{\partial P_t} q\cdot n\, dA_y = \int_{P_t} \operatorname{div} q\, dV_y$$

Reduced transport

$$\forall P_t \qquad \int_{P_t}\left(\rho \frac{De}{Dt} + \operatorname{div} q - T:D - Q\right) dV_y = 0$$

$\longrightarrow$ First law of thermodynamics (balance of energy) (I)

$$\boxed{\rho\frac{De}{Dt} = T:D + \underbrace{Q - \operatorname{div} q}}$$

Continuum version of it

Continuum version of it

New terms relative to what we did for hyperelastic material

2nd law of thermodynamics:

$$W + Q = \dot{U}$$

External power

rate of heat

energy rate

1st law

$$\frac{Q}{\Theta} < \dot{S}$$

absolute temperature

entropy

2nd law

How about continuum version of 2nd law of thermodynamics

η: specific Entropy

Entropy supply: interior flux, boundary flux

$$\int_{P_t} \rho \eta_b \, dV_y + \int_{\partial P_t} \eta_s \, dA_y$$

Entropy supply

$$\int_{P_t} \frac{Q}{\Theta} dV_y - \int_{\partial P_t} \frac{q \cdot n}{\Theta} dA_y$$

$$S = \int \rho \eta \, dV_y$$

Total entropy

Specific entropy

2n law

entropy production $\leq \underbrace{\frac{D}{Dt} S}_{\dot{S}}$ material rate

The rate of entropy flux (q) and entropy supply (Q) cannot exceed the rate of increase of entropy

$$\int_{P_t} \frac{Q}{\Theta} dV_y - \int_{\partial P_t} \frac{q \cdot n}{\Theta} dA_y \leq \frac{D}{Dt} \int \rho \eta \, dV_y$$

reduced transport

P_t $\quad\downarrow\quad$ ∂P_t $\qquad$ reduced transport

$$-\int_{P_t} \operatorname{div}\left(\frac{q}{\theta}\right) dV_y \leq \int_{P_t} \rho \dot{\eta}\, dV_y$$

$\longrightarrow$
$$\int_{P_t} \left\{ \rho\dot{\eta} - \frac{Q}{\theta} + \operatorname{div}\left(\frac{q}{\theta}\right) \right\} dV_y \geq 0$$

$\longrightarrow$
$$\boxed{\rho\gamma = \rho\dot{\eta} - \frac{Q}{\theta} + \operatorname{div}\left(\frac{q}{\theta}\right) \geq 0}$$

specific entropy production

2nd law of thermodynamics, Clausius-Duhem inequality

This equation is used to narrow down the form of constitutive equation for solids (e.g. damage mechanics and plasticity) and fluids to derive the so-called thermodynamically consistent constitutive equations.

Other ways to define energy density besides e:

1. Helmholtz free energy per unit mass: $\psi = e - \eta\theta$ (specific energy density)
2. Enthalpy per unit mass
3. Gibbs free energy

See Abeyaratne equation (5.15)

The full set of equations:

Full $\frac{\partial}{\partial t}$ & div operator

$$\frac{D\rho}{Dt} + \rho \operatorname{div} v \Rightarrow \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho v) = 0 \quad \text{b. mass}$$

$$\rho\frac{Dv}{Dt} + \operatorname{div} T + \rho b = 0 \Rightarrow \frac{\partial \rho v}{\partial t} + \operatorname{div}(\rho v \otimes v - T) = \rho b \quad \text{linear momentum}$$

$$T = T^T \quad \text{angular momentum}$$

$$\rho\dot{e} - T:D - \operatorname{div} q + Q = 0 \rightarrow \frac{\partial \rho e}{\partial t} + \operatorname{div}(q + \rho e v) = T:D + Q$$

energy balance (1st law thermo)

$$\frac{Q}{\theta} - \operatorname{div}\left(\frac{q}{\theta}\right) \leq \rho\dot{\eta} \rightarrow \frac{Q}{\theta} - \operatorname{div}\left(\frac{q}{\theta}\right) \leq \frac{\partial \rho\eta}{\partial t} + \operatorname{div}(\rho\eta v)$$

2nd law of thermo

2nd law of thermo

Another way to write 2nd law
Q is replaced from 1st law
$\psi = e - \eta\theta$

$$\boxed{\rho\dot{\psi} + \rho\eta\dot{\theta} - T:D + \frac{\mathrm{grad}\,\theta\cdot q}{\theta} \leq 0}\quad \mathrm{II}b$$

Example 1: Constitutive equation for COMPRESSIBLE ELASTIC FLUID (12.1 Abeyaratne)

$$T = G(F) \xrightarrow{\text{objectivity}} T = G(C) \;-$$

Symmetry group for compressible elastic fluids contains ALL unimodal transformations

Refer to

Definition 112 (Noll, 1958) *Given an elastic body and a reference configuration that corresponds to the region* $\overset{0}{\mathcal{B}}$, *the* material symmetry group *at the material point identified by* $\mathbf{x}$ *in the reference configuration is the set*

$$\underset{\mathbf{x}}{\mathrm{Msg}} = \{\mathbf{H} \in \mathrm{Unim}\ \mathcal{V}^+ : \mathbf{G}(\mathbf{FH}, \mathbf{x}) = \mathbf{G}(\mathbf{F}, \mathbf{x})\ \forall\ \mathbf{F} \in \mathrm{Lin}\ \mathcal{V}^+\}. \longrightarrow \nabla q$$

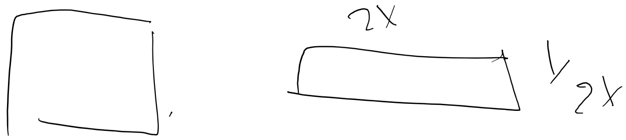
Again, it should be emphasized that the material symmetry group is characterized by tensors $\mathbf{H}$ that correspond to the gradients at $\mathbf{x}$ of deformations — not the deformations themselves. This is because the mass density and the elastic response function in the second reference configuration depend only on the gradient of the connecting deformation. Also, note that $\mathbf{H} \in \mathrm{Msg}_{\mathbf{x}}$ is not a tensor field, but rather the value of a tensor field at $\mathbf{x}$.

The following theorem presents a property of all orthogonal elements of $\mathrm{Msg}_{\mathbf{x}}$ that derives from the Principle of Material Frame-Indifference.

For isotropic solid material, what should be symmetry group of constitutive equation?

All rotations belong to symmetry group.

What is the material that ALL unimodal tensors belong to its symmetry group?



"Elastic fluids" would have all unimodal tensors in their const. eqn. symmetry group:

inviscid

$$G(FH) = G(F) \quad \forall H \ni \det H = 1$$

on can show because of

G(F) reduces to simple relation

$$G(F) = \bar{G}(\underbrace{\det F}_{J}) = \bar{G}\left(\frac{\rho}{\rho_0}\right) = \bar{G}(\rho)$$

$$F \to C \to \det C \quad \left(\det F = \frac{\rho}{\rho_0}\right.$$

$$T = G(\rho)$$

Add thermal effects

Constitutive equations for compressible elastic fluid

$$g = \operatorname{grad}\theta$$

$$T = \hat{T}(\rho, \theta, \operatorname{grad}\theta)$$

$$\psi = \hat{\psi}(\rho, \theta, \operatorname{grad}\theta)$$

$$\eta = \hat{\eta}(\quad // \quad)$$

$$q = \hat{q}(\quad // \quad)$$

equipresence

IIb

$$\rho\dot{\psi} + \rho\eta\dot{\theta} - T:D + \frac{\operatorname{grad}\theta \cdot q}{\theta} \le 0$$

$$\dot{\psi} = \underbrace{\frac{\partial\psi}{\partial\rho}}_{\psi_\rho}\dot{\rho} + \underbrace{\frac{\partial\psi}{\partial\theta}}_{\psi_\theta}\dot{\theta} + \underbrace{\frac{\partial\psi}{\partial g}}_{\psi_g}\dot{g}$$

$$D = \operatorname{sym} L$$

Balance of mass

$$\dot{\rho} + \rho\,\mathrm{div}\,v = 0 \longrightarrow \dot{\rho} = -\rho\,\mathrm{div}\,v = -\rho(D:I)$$

$$\rho(-\rho(D:I))\psi_\rho + \rho\psi_\theta\dot{\theta} + \rho\psi_g\dot{g} - T:D + \rho\eta\dot{\theta} + gq/\theta$$

$$+\frac{g\cdot q}{\theta} \leq 0$$

$$\boxed{D:\left(T+\rho^2\psi_\rho I\right)+\underbrace{\left(\rho\eta+\rho\psi_\theta\right)\dot{\theta}}+\rho\psi_g\dot{g}+\frac{g\cdot q}{\theta}\leq 0}$$

$D = \text{Sym} L = \text{Sym} \nabla_y v$

$\dot{\theta}$ = temperature rate

$\dot{g}$ = rate of grad θ

$D, \dot{\theta}, \dot{g}$ arbitrary $\longrightarrow$ using 2nd law

$$\begin{cases} T+\rho^2\psi_\rho I = 0 & a \\ \rho(\eta+\psi_\theta) = 0 & b \\ \rho\psi_g = 0 & c \\ \frac{g\cdot q}{\theta} \leq 0 & d \end{cases}$$

$$T = -\rho^2\psi_\rho I = -\rho^2\psi_\rho \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ no shear stress!

$$\eta = -\psi_\theta$$

$\psi_g = 0 \longrightarrow \psi$ does not depend on g & from a, b so do η & T

Summary

$$\psi(\rho, \theta)$$

$$\longrightarrow T = -pI \qquad p = \rho^2\psi_\rho$$

$$\eta = -\psi_\theta$$

$g\cdot q \leq 0 \longrightarrow$ example $q = -K(\rho, \theta, |g|)\,g$ Fourier heat law

$$\boxed{g \cdot q \leq 0 \longrightarrow \text{example} \quad q = -\underbrace{K(\rho, \Theta, |g|)}_{\text{2nd order sym tensor}}\, g \quad g = \operatorname{grad}\Theta}$$

Fourier heat law satisfies it

Perfect gas: Eqns for $p(\rho,\Theta)$ & $e(\rho,\Theta)$ instead of $\psi(\rho,\Theta)$

$\psi = ?$

$$\boxed{p(\rho,\Theta) = \underbrace{R}_{\text{specific gas constant}}\rho\Theta \qquad e = \underbrace{c}_{\text{volumetric heat capacity}}\Theta}$$

$$p = R\rho\Theta = \rho^2\psi_{,\rho} \longrightarrow \psi_{,\rho} = \frac{R\Theta}{\rho} \longrightarrow$$

$$\boxed{\psi = R\Theta \ln\rho + \underbrace{f(\Theta)}_{\text{constant of integration.}}}$$

$e = \psi + \eta\Theta$, $\boxed{\eta = -\psi_{,\Theta}}$

$\longrightarrow \quad c\Theta = R\Theta \ln\rho + f(\Theta) - (R\ln\rho + f'(\Theta))\,\Theta$

$\longrightarrow \quad \boxed{c\Theta = f(\Theta) - f'(\Theta)\Theta}$ ODE for Θ note $\left(\frac{f(\Theta)}{\Theta}\right)' = \frac{f'\Theta - f}{\Theta^2}$

$\longrightarrow$ Find $f(\Theta)$ from the ODE

$$\boxed{\psi(\rho,\Theta) = R\Theta\ln\rho - c\Theta\ln\left(\frac{\Theta}{\Theta_0}\right) + c\Theta}$$