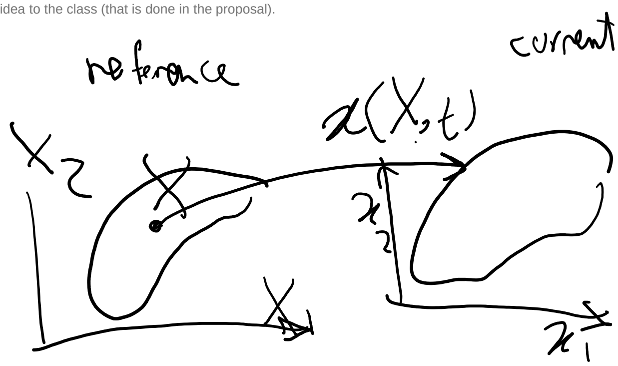
CM2021/08/19

Thursday, August 19, 2021 2:35 PM

Syllabus:
http://www.rezaabedi.com/wp-content/uploads/Courses/ContinuumMechanics/ContinuumMechanicsSyllabus.pdf

7 Homework assignments
1 final exam (take-home ?)
1 term project

- 1) An up to 4 pages paper/proposal(including references if any) on a topic related to continuum mechanics. The format of the document is either that of a
  - **Research article** mostly focusing on introducing a topic of interest and presenting related results. Suggested sections are abstract, introduct ion, formulation, results (can present results from existing literature, doesn't need to be from your own research), conclusion.
  - **Research proposal** that basically introduces a problem, discusses current state of the art and research gaps, and finally proposes a new approac h to address the mentioned research gaps. Suggested sections are (abstract), introduction (why this problem is important and what is the main contribution of the proposed work), background (state of the art and what are the existing gaps and challenges), objective (describing the goal and objectives of the research), research tasks (what is propos ed to be done). Some optional sections are intellectual merits and broader impacts as often required in research proposals.
- 2) Presentation of the article on the "Presentation day". Each student will have about 15 minutes to present the material in the article (and related to it) to the entire class.
- Notes:
  - The choice between research article or proposal is up to the student. The topic can be related to your own research work (as long as it is related to continuum mechanics) or any other topic related to the course that is of interest to you. I can help you in choosing a topic if needed. Please confirm your research topic by the end of ***10/30/2020***. *Some proposed topics are:*
    - Mathematical background:
      - Vectors vs. covectors, tensors and cotensors / differential form notation.
      - Curvilinear and non orthonormal coordinate systems.
    - Kinematics:
      - Eulerian versus Lagrangian strains.
      - Arbitrary Lagrangian Eulerian (ALE) formulations.
      - Objective rates of deformation.
    - Balance laws, forces / stress:
      - Balance laws in spacetime.
      - Jump condition (Rankine-Hugoniot jump conditions); shocks, expansion waves, contact discontinuity.
      - Thermodynamic laws (in relation to the course content).
    - Constitutive Equations (possibly in combination with kinematics / balance laws):
      - Constitutive equations for various types of fluids.
      - Gradient elasticity theory (formulations that use beyond strain value in the constitutive equation) – topic for solid mechanics.
      - Thermodyanmically motivated damage / phase field models for solid materials.
      - Constitutive equations (and if needed kinematics / balance laws) for specific group of materials:
        - Dispersive materials: viscoelasticity, dynamic metamaterials, *etc*.
        - Any other type of so-called mechanical metamaterials (light weight, auxetic, pentamode, origami, *etc*.).
        - 3D printed materials.
        - Granular materials.
        - Foams, soft material, etc..
- If you choose the proposal format, your presentation will be on the general topic of your proposal not actually on selling your idea to the class (that is done in the proposal).



Course outline:
1. Mathematical preliminaries:
   a. Indicial notation & summation convention
   b. Vector notation
   c. Tensors
   d. Coordinate transformation
   e. Derivative operations (curl, grad, div)
2. Kinematics: Velocity, strain, strain gradient, etc.
3. Force-like fields: stress tensor
4. Balance laws: Balance of linear momentum (sum of forces = 0 static or … in dynamic), balance of energy, balance of mass.
5. Constitutive equations

strain → Stress
deformation rate ↗

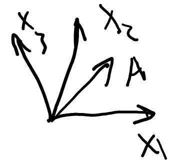
Heat eqn $\underset{\text{heat flux}}{q} = -k \nabla T$ Fourier heat conduction

---

Indicial notation:

$A = (A_1, A_2, A_3)$

It's often assumed $d = 3$



d = dimension

$A_1 > 0$, $A_2 > 0$, $A_3 > 0$

$A_i > 0$ for $\underbrace{i \in \{1, 2, 3\}}_{\text{what if we drop this?}}$

$\boxed{A_i > 0}$

free index. It takes any of the values 1 to 3.

Expressions we deal with are additions / subtractions of terms multiplying each other (or their function values)

$$a_i + A_{ij} b_j + C_{i2} d_2 = \frac{\partial d_{ik}}{\partial x_k} + \cos(e_i) + \cancel{z} + 5$$

$\sum_{j=1}^{3} A_{ij} b_j$ j dummy index

There is just 1 i for all these terms. That means this expression holds for i in {1, 2, 3}.

For example, the following does not make sense:

$\cancel{a_i + b_j > 0}$ $\quad$ $a_1 + b_1 > 0$ $\quad$ $\cancel{a_1 + b_2 > c}$

$a_2 + b_1 > 0$

matrices

$a = A \quad B$

$$C = A + B$$ matrices

3x3

$$C_{ij} = A_{ij} + B_{ij}$$ both are free

$C_{11} = A_{11} + B_{11}$, $C_{12} = A_{12} + B_{12}$, —

$C_{21} = A_{21} + B_{21}$

$i \to 1 \ldots 3$
$j \to 1 \ldots 3$ } 9 eqns

$C_{33} = \ldots$

$\vec{a} = \vec{b}$ $\quad a_i = b_i \quad$ or $\quad a_j = b_j$

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

scalar

$$a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3 = a^T b$$

inner product

$$[a_1 \ a_2 \ a_3] \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

1x3 3x1

$$a \otimes b = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} [b_1 \ b_2 \ b_3] = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & & \\ & — & a_3 b_3 \end{bmatrix}$$

$= ab^T$

3x1 1x3

dyadic product

9 eqns

$$(a \otimes b)_{11} = a_1 b_1$$
$$(a \otimes b)_{12} = a_1 b_2$$
$$\vdots$$
$$(a \otimes b)_{21} = a_2 b_1$$
$$\vdots$$

$$(a \otimes b)_{ij} = a_i b_j$$

$$a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3$$
$$= \sum_{i=1}^{3} a_i b_i$$
$$:= a_i b_i$$

2 times repeated index imply summation

$$a \cdot b = a_i b_i = a_k b_k$$

i dummy index

2nd Example

$$C_{ij}$$

$$\underbrace{C_{ii}}_{\text{repeated}} = C_{11} + C_{22} + C_{33} \quad \text{called trace}(C)$$

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

write an expression that says all the diagonals are positive

$$\cancel{C_{ii} > 0} \equiv \cancel{\sum_{i=1}^{3} C_{ii} > 0}$$

$$C_{ii} > 0 \quad \forall i \in \{1, 2, 3\} \quad ✓$$

other ways people do it

$C_{ii} > 0$ (no summation convention)

or $C_{i\underline{i}} > 0$

or $C_{i(i)} > 0$

→ they define $\underline{\ \ }$, ( ) as no summation convention.

$b = \underset{\sim}{A} a$ vector

vector matrix

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$b_1 = A_{11} a_1 + A_{12} a_2 + A_{13} a_3 \rightarrow b_1 = A_{1j} a_j$

$b_2 = A_{21} a_1 + A_{22} a_2 + A_{23} a_3$ $\quad b_2 = A_{2j} a_j$

$b_3 = A_{31} a_1 + A_{32} a_2 + A_{33} a_3$ $\quad b_3 = A_{3j} a_j$

$$\boxed{a_i = A_{ij} b_j}$$

$b = Aa$, $c = Bb$ — write c in terms of a

$c = Bb = B(Aa) = \underbrace{(BA)}_{C} a$

$C = BA$

$a_i = C_{ij} a_j$

Indicial expression of this

$b_i = A_{ij} a_j$ (2d)

$c_i = B_{ij} b_j$ (2d)

$i \rightarrow j$

b_i we need b_i (2d)

$\cancel{b_i = A_{ij} a_j}$

$\cancel{b_j = A_{jj} a_j}$ more than 2 j

$\cancel{c_i = B_{ij} A_{jj} a_j}$

$b_i = A_{ik} a_k$ now can change $i \rightarrow k$

$b_j = A_{jk} a_k$ ✓

$$C_i = B_{ij} b_j$$

$$\boxed{C_i = B_{ij} A_{jk} a_k}$$

$$C = (BA)\, a$$

$$C_i = (BA)_{ik} a_k$$

$$\boxed{(BA)_{ik} = B_{ij} A_{jk}}$$

$(BA)_{ij}$ = row i of B · column j of A

$$\begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$(BA)_{32} = B_{31} A_{12} + B_{32} A_{22} + B_{33} A_{32}$$

Exceptions:

$C_{i\underline{i}}$ (no summation)

Eigen decomposition

$A\underset{\text{eigenvector}}{u} = \underset{\text{eigenvalue}}{\lambda} u$ $\qquad A_{3\times3}$

A symmetric (3×3) → eigen #

$\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}$ are real

can find vhes $u^{(1)}, u^{(2)}, u^{(3)}$ orthonormal

$$u^{(i)} \cdot u^{(j)} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

$u^{(1)} \quad u^{(2)}$

can find three $u^{\{1\}}$ $u^{\{2\}}$ $u^{\{3\}}$ orthonormal

$u^{\{i\}} \cdot u^{\{j\}} = \begin{cases} 1 & i=j \end{cases}$

$u^{\{3\}}$ $u^{\{2\}}$ $u^{\{1\}}$

$$A \underbrace{\left[u^{\{1\}} \middle| u^{\{2\}} \middle| u^{\{3\}} \right]}_{U} = \left[u^{\{1\}} \middle| u^{\{2\}} \middle| u^{\{3\}} \right] \underbrace{\begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}}_{\text{diagonal } \Lambda}$$

$A u^{\{i\}} = \underbrace{\lambda^i u^{\{i\}}}$ no summation

$$\underset{3\times3}{A}\,\underset{3\times3}{U} = \underset{3\times3}{U}\,\underset{3\times3}{\Lambda}$$

$$A = U \Lambda U^{-1}$$

U orthogonal

$$U^{-1} = U^T$$

$$\boxed{A = U \Lambda U^T}$$ if A is symmetric

$$A = \left[u^{\{1\}} \middle| u^{\{2\}} \middle| u^{\{3\}} \right] \begin{bmatrix} \lambda^{\{1\}} & & \\ & \lambda^{\{2\}} & \\ & & \lambda^{\{3\}} \end{bmatrix} \begin{bmatrix} u^{\{1\}} \\ \hline u^{\{2\}} \\ \hline u^{\{3\}} \end{bmatrix}$$

$a \otimes b = a b^T = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} [b_1 \; b_2 \; b_3]$

$$A = \lambda^1 u^{\{1\}} \otimes u^{\{1\}} + \lambda^2 u^{\{2\}} \otimes u^{\{2\}} + \lambda^3 u^{\{3\}} \otimes u^{\{3\}}$$

$$\boxed{A = \sum_{i=1}^{3} \lambda^i u^{\{i\}} \otimes u^{\{i\}}}$$

can we write it as

$$A = \lambda^i u^{\{i\}} u^{\{i\}}$$

cause repeated i imply summation?

only 2 repeated index means summation

only 2 repeated index means summation

$$\boxed{A = \sum_{i=1}^{3} \lambda^{i} u^{\{i\}} \otimes u^{\{i\}}}$$ ✓

$$A u^{\{i\}} = \lambda^{i} u^{\{i\}}$$ in correct indicial notation

$$A u^{\{i\}} = \lambda^{i} u^{\{\underline{i}\}}$$

or $$A u^{\{i\}} = \lambda^{i} u^{\{i\}}$$ no summation on i