CM2021/08/24

Tuesday, August 24, 2021 4:25 PM

Kronecker's delta:

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \qquad \delta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

So, it's basically the identity matrix

Some properties of δ_{ij}

1. $\delta_{\underline{ii}} = 1$ no summation

2. $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$

3. $\delta_{ij} b_j = \delta_{i1} b_1 + \delta_{i2} b_2 + \delta_{i3} b_3 = b_i$

$i=1 \quad \overset{1}{\delta_{11}} b_1 + \cancel{\delta_{12}}^{0} b_2 + \cancel{\delta_{13}}^{0} b_3 = b_1$

$i=2 \quad \cancel{\delta_{21}}^{0} b_1 + \overset{1}{\delta_{22}} b_2 + \cancel{\delta_{23}}^{0} b_3 = b_2$

$i=1 \rightarrow b_1$
$i=2 \quad b_2$
$i=3 \rightarrow b_3$

b_i

Matrix form says $\vec{b} = \bar{\bar{I}} \vec{b}$

$$\boxed{\delta_{ij} b_j = b_i}$$

replace this with

$$\delta_{ij} C_{kl\,ij\,mp} = \underline{C_{kl\,i\,i\,mp}} := \sum_{i=1}^{3} C_{kl\,ii\,mp}$$

meaning

$$\delta_{ij} C_{kl\,i\,q\,mp} = C_{kl\,j\,q\,mp}$$

replace with

$$\delta_{ij} C_{kl\,j\,q\,mp} = C_{kl\,i\,q\,mp}$$

replace with

$$c = Aa \quad + 5a$$

$$c_i = A_{ij} a_j + 5 a_i \rightarrow a_j \qquad a_j = ?\, a_i \quad (\delta_{ij})$$

$$= A_{ij} a_j + 5\delta_{ij} a_j$$

$$c_i = (A_{ij} + 5\delta_{ij})\, a_j \qquad c = (A + 5I)\, b$$

Permutation or Alternating Symbol

$$\epsilon_{ijk} \longrightarrow 27 \text{ combinations}$$

$i \to 3$, $j \to 3$, $k \to 3$

Definition: $i = j$ or $j = k$ or $i = k$ (at least two indices are equal) $\} \rightarrow \epsilon_{ijk} = 0$

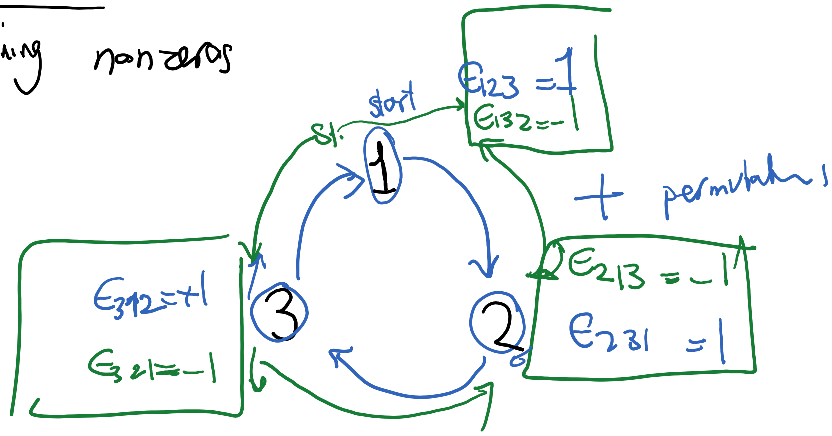
Examples:

$\epsilon_{112} = 0$ $\quad$ $\epsilon_{111} = 0$

$\epsilon_{122} = 0$ $\quad$ $\cancel{\epsilon_{123} = 0}$

Only 6 remaining nonzeros

− permutations



Another way to get these 6 values is to count how many permutations are needed relative to ϵ_{123}

- Even number -> +1
- Odd number -> -1

$$\epsilon_{jik} = -1\, \epsilon_{ijk}$$

$$\epsilon_{jki} = (+1)\, \epsilon_{ijk}$$

even: ① $\epsilon_{ijk} \downarrow \epsilon_{jik}$ ② $\downarrow \epsilon_{jki}$

	start with ϵ_{123}		#	create this
ϵ_{123}	ϵ_{123}		0	+1
ϵ_{231}	ϵ_{123}	$\rightarrow \epsilon_{213} \rightarrow \epsilon_{231}$	2	+1
ϵ_{312}	ϵ_{123}	$\rightarrow \epsilon_{321} \rightarrow \epsilon_{312}$	2	+1
ϵ_{132}	ϵ_{123}	$\rightarrow \epsilon_{132}$	1	−1
ϵ_{321}	ϵ_{123}	$\rightarrow \epsilon_{321}$	1	−1
ϵ_{213}	ϵ_{123}	ϵ_{213}	1	−1

Use of alternating symbol

- Determinant of a matrix $\quad$ det A
- Inverse of a matrix
- Cross product, curl $\quad$ $\vec{a} \times \vec{b}$ $\quad$ $\nabla \times \vec{V}$

Determinant

$A = (A_{11})$

$\det A = A_{11}$

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$$

$$\det A = A_{11} \det(A_{22})(-1)^{1+1} + (-1)^{1+2} A_{12} \det A_{21}$$

$$A_{11}A_{22} - A_{12}A_{21}$$

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$\det A = (-1)^{1+1} A_{11} \det \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} + (-1)^{1+2} A_{12} \det \begin{bmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{bmatrix}$$
$$+ (-1)^{1+3} A_{13} \det \begin{bmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix}$$

$$\boxed{\det A = A_{11}A_{22}A_{33} - A_{11}A_{23}A_{32} - A_{12}A_{21}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32} - A_{13}A_{22}A_{31}}$$

Claim $\det A = \epsilon_{ijk} A_{1i} A_{2j} A_{3k}$

$$= \underset{+1}{\epsilon_{123}} A_{11}A_{22}A_{33} + \overset{-1}{\epsilon_{132}} A_{11}A_{23}A_{32} + \overset{-1}{\epsilon_{213}} A_{12}A_{21}A_{33} + \overset{+1}{\epsilon_{231}} A_{12}A_{23}A_{31}$$
$$+ \underset{+1}{\epsilon_{312}} A_{13}A_{21}A_{32} + \underset{-1}{\epsilon_{321}} A_{13}A_{22}A_{31} + 21 \text{ zero terms}$$

eg $\epsilon_{112} A_{11} A_{21} A_{32}$

0

☒

$$\boxed{\det A = \epsilon_{ijk} A_{1i} A_{2j} A_{3k}}$$

$$\det A = \epsilon_{ijk} A_{i1} A_{j2} A_{k3}$$

back to this

In HW1 you'll show

$$\epsilon_{mnp} \det A = \epsilon_{ijk} A_{im} A_{jn} A_{kp}$$

$m=1$
$n=2$
$p=3$

Show for any two indices the same, you'll get zero on the RHS

$$\epsilon_{112} \det A = \epsilon_{ijk} A_{i1} A_{j1} A_{kp}$$
$$= -\epsilon_{jik} A_{ji'} A_{i'1} A_{kp}$$

& for other 5 cases of $m \neq n$, $n \neq p$, $p \neq m$ turn ϵ_{mnp} to ϵ_{123}

If you show it for these cases it's fine

Some properties of determinant

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, B = \ldots$$

- $C = \begin{bmatrix} 0 & 0 & 0 \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$ $\det C = 0$

$$\det C = \epsilon_{ijk} C_{1i} C_{2j} C_{3k}$$
$$\epsilon_{ijk}\, 0\, A_{2j}\, A_{3k} = 0$$

$$C = \begin{bmatrix} A_{21} & A_{22} & A_{23} \\ A_{11} & A_{12} & A_{13} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$A = \begin{bmatrix} r_1^A \\ r_2^A \\ r_3^A \end{bmatrix}$$

$$\det C = -1 \det A$$

$\det C = -\det A$

$$\det C = \epsilon_{ijk} C_{1i} C_{2j} C_{3k}$$

$$= \epsilon_{ijk} A_{2i} A_{1j} A_{3k}$$

$$= (-1)\,\epsilon_{jik} A_{1j} A_{2i} A_{3k} = -\det A$$

$$C = \begin{bmatrix} \lambda A_{11} & \lambda A_{12} & \lambda A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$\det C = \lambda \det A$$

$C = \lambda A$

$\det C = \lambda^n \det A$

for $n \times n$ matrix

proof

$$\det C = \epsilon_{ijk} C_{1i} C_{2j} C_{3k}$$

$$= \epsilon_{ijk} (\lambda A_{1i}) A_{2j} A_{3k}$$

$$= \lambda \left[\epsilon_{ijk} A_{1i} A_{2j} A_{3k}\right] = \underline{\lambda \det A}$$

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{ijk}\, \lambda\, A_{1i} A_{2j} A_{3k}$$

constant

$$\lambda \underbrace{\sum\sum\sum \epsilon_{ijk} A_{1i} A_{2j} A_{3k}}_{\det A}$$

$$C = \begin{bmatrix} A_{11}+B_{11} & A_{12}+B_{12} & A_{13}+B_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

r_1 of $C = \text{row}_1(A) + \text{row}_1(B)$

expanded

$$\det C = \det \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ & & \\ & & A_{33} \end{bmatrix} + \det \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$\det C = \epsilon_{ijk} C_{1i} C_{2j} C_{3k}$$

$\det C = \epsilon_{ijk}\, C_{1i}\, C_{2j}\, C_{3k}$

$\epsilon_{ijk}(A_{1i}+B_{1i})\, A_{2j}\, A_{3k} = \epsilon_{ijk} A_{1i} A_{2j} A_{3k} + \epsilon_{ijk} B_{1i} A_{2j} A_{3k}$

$C = A + B \qquad \det C \neq \det A + \det B$

$C = AB \qquad \det C = \det A \det B$ ✓ HW

Some indicial notation examples:

scalar $Q = x^T A x \qquad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$= [x_1 \; \cdots \; x_n] \begin{bmatrix} A_{11} & \cdots \\ & \end{bmatrix}_{n\times n} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad A_{n\times n}$ matrix

$Q = (x^T)_i\, (Ax)_i$ like energy

$= x_i\, A_{ij} x_j$

$$\boxed{Q = x_i A_{ij} x_j}$$

$$\frac{\partial Q}{\partial x_i} = \frac{\partial (x_i A_{ij} x_j)}{\partial x_i} \quad \times \quad \text{3 repeated } i\text{'s}$$

$\rightarrow$ force of x_i

$= A_{ij} x_j \quad \times$

$$\frac{\partial Q}{\partial x_k} = \frac{\partial x_i A_{ij} x_j}{\partial x_k} = \underbrace{\frac{\partial x_i}{\partial x_k}}_{=\delta_{ik}} A_{ij} x_j$$

$$\frac{\partial Q}{\partial x_k} = \frac{\partial x_i A_{ij} x_j}{\partial x_k} = \frac{\partial x_i}{\partial x_k} A_{ij} x_j + x_i \frac{\partial A_{ij}}{\partial x_k} x_j + x_i A_{ij} \frac{\partial x_j}{\partial x_k}$$

0

δ_{jk}

$$= \boxed{\delta_{ik} A_{ij} x_j} + x_i A_{ij} \delta_{jk}$$

$$= A_{kj} x_j + x_i A_{ik} = (A_{kj} + A_{jk}) x_j$$

change $i \to j$

$$\boxed{\frac{\partial Q}{\partial x_k} = (A_{kj} + A_{jk}) x_j}$$

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & & \\ & & A_{33} \end{bmatrix} \qquad A^T = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & & \\ & & \end{bmatrix}$$

$$\frac{A + A^T}{2} = \begin{bmatrix} A_{11} & \frac{A_{12}+A_{21}}{2} & 1,3 \\ \frac{A_{12}+A_{21}}{2} & A_{22} & 2,3 \\ & & A_{33} \end{bmatrix}$$

$$\frac{\partial Q}{\partial x_k} = 2\left(\frac{A + A^T}{2}\right)_{ij} x_j \qquad \text{sym}A = \frac{A + A^T}{2}$$

$$\boxed{\frac{\partial Q}{\partial x} = 2\left(\frac{A + A^T}{2}\right) x}$$

$$\boxed{\frac{\partial Q}{\partial x} = 2(\text{sym}A)\, x}$$

if A is sym $(A = A^T)$

$\partial Q / \partial x = 2Ax$