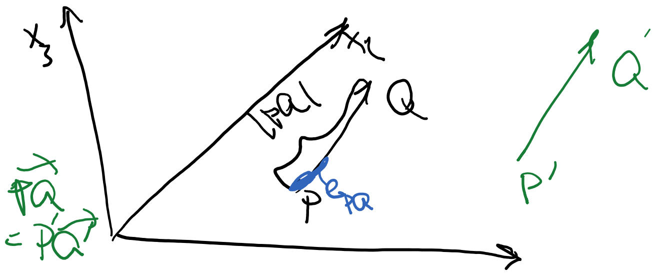
CM2021/08/26

Thursday, August 26, 2021 4:17 PM

Vector spaces



Vector PQ is identified by 3 things:

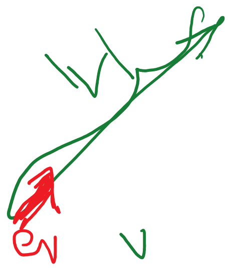
1. Length of PQ: $|PQ|$
2. Direction of PQ: e_{PQ}

] often $\vec{PQ} = \vec{P'Q'}$

e_{PQ} is of length 1 & direction of PQ

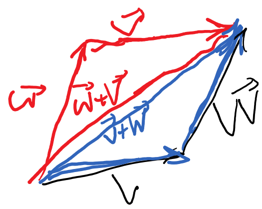
3. Sometimes we also care about base point P (often we don't care about this). If we differentiate vectors by their base point as well, they are called bound vectors

non-bound vector



$$e_v = \frac{\vec{v}}{|v|}$$

Addition



$$\vec{w}+\vec{v} = \vec{v}+\vec{w}$$

Scalar product

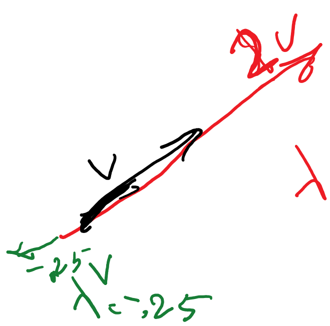
$\lambda \in \mathbb{R}$

$\vec{v}$

$\lambda \vec{v}$:

$$|\lambda \vec{v}| = |\lambda||v|$$

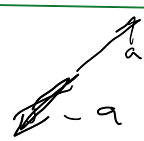
$$e_{\lambda\vec{v}} = \begin{cases} e_v & \lambda > 0 \\ -e_v & \lambda < 0 \\ 0 & \lambda = 0 \end{cases}$$



$\lambda = 2$

$\lambda = -2.5$

$$-a := (-1)a$$



$$(-a)+a = a+(-a) = 0$$

$$a - b = a + (-b)$$

$a - b = a + (-b)$

Properties of vectors

A) A1) $a + b = b + a$ Commutative Addition

A2) $a + (b + c) = (\vec{a} + \vec{b}) + \vec{c}$ associative

A3) $a + 0 = 0 + a = a$

0 → neutral member w.r.t addition

need to be proved for any vector space

$\boxed{\text{Proved}}$

$\forall a \; \exists (-a) \ni a + (-a) = (-a) + a = 0$

Vector space is an Abelian group w.r.t +

it's not added as an axiom of vector space

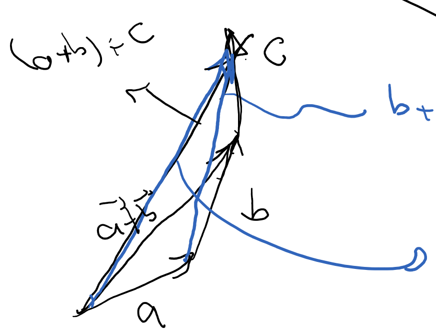
B) Scalar properties $\lambda, \mu \in \mathbb{R}$, a, b vectors

B1) $(\mu\lambda)a = \mu(\lambda a)$ scalar product property

B2) $(\lambda + \mu)a = \lambda a + \mu a$ distributive with respect to scalar addition

B3) $\lambda(a + b) = \lambda a + \lambda b$ " " " vector addition

B4) $1 \cdot a = a$



equal to $a + (b + c)$

show this

$$\boxed{0 \cdot a = 0}$$

scalar zero — zero vector

$$a + 0 \cdot a = \underbrace{1 \cdot a}_{B4} + 0 \cdot a = (1+0) \cdot a \quad B2$$

must be zero

$a = 1 \cdot a$

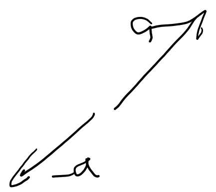
$1 \cdot a = a$ B4

why

$$a + (-a) = 0$$

det.

$$\underset{B4}{1 \cdot a} + (-1)\, a = (1 + (-1))\, a = 0 \cdot a = 0$$

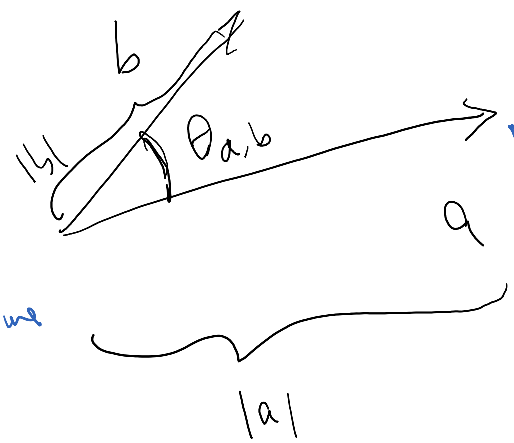


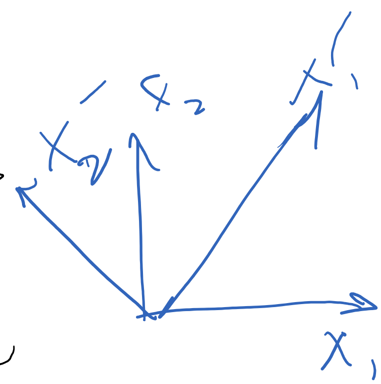
Inner product of vectors

if $a = 0$ or $b = 0$ $a \cdot b = 0$

$$\vec{a} \cdot \vec{b} = |a||b| \cos\theta_{a,b}$$

good definition
for any coordinate system
(defined later) we get the same value





We use inner product to compute the project of b to a or a to b.
In general we use inner product for projection operators.

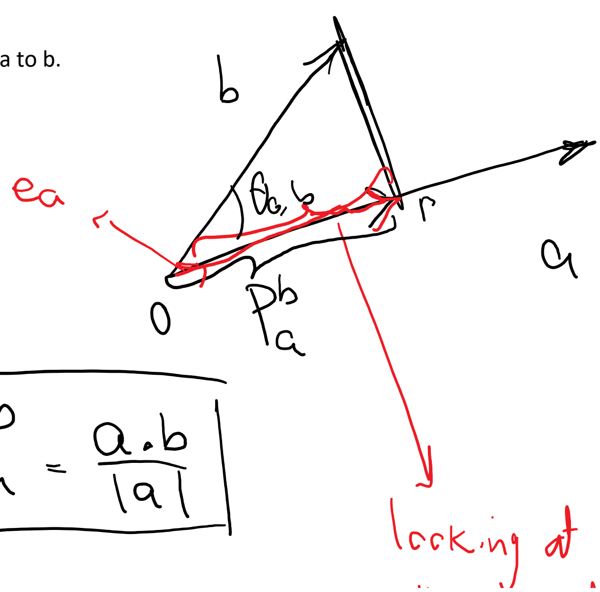
$$P_a^b = |b| \cos\theta_{a,b}$$

$$= \frac{|a||b| \cos\theta_{a,b}}{|a|}$$

$$= \frac{a \cdot b}{|a|}$$

$$\boxed{P_a^b = \frac{a \cdot b}{|a|}}$$

scalar



this a vector of length p_a^b oriented in a

$$\vec{P}_a^b = P_a^b \, e_a = \frac{a\cdot b}{|a|} \circ \frac{a}{|a|}$$

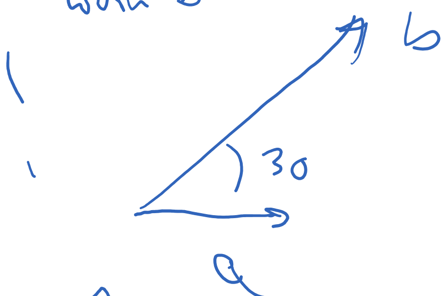
$$\boxed{\vec{P}_a^b = \left(\frac{a\cdot b}{a\cdot a}\right)a = \left(\frac{a\cdot b}{|a|^2}\right)a}$$

A few examples:

1) $\theta_{a,b} = 0$

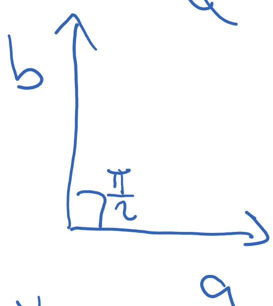
$$a\cdot b = |a||b|\underbrace{\cos\theta_{a,b}}_{1} = |a||b|$$

a is aligned with b



$a\cdot b$ gets smaller

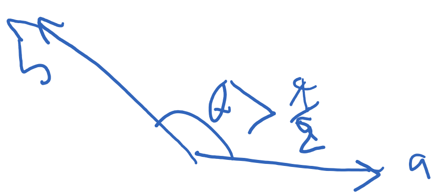
2) $\theta = \frac{\pi}{2}$



$$a\cdot b = |a||b|\underbrace{\cos\tfrac{\pi}{2}}_{0} = 0$$

$$\boxed{\theta = \frac{\pi}{2} \iff a\cdot b = 0}$$

$a \perp b$



$a\cdot b < 0$

$\cos\theta < 0$

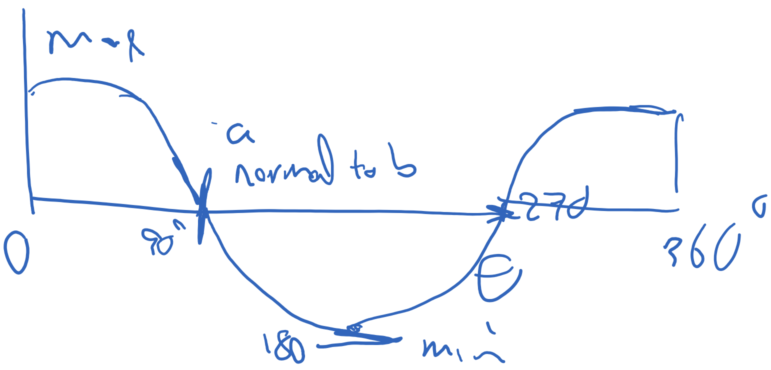
3) $\theta = \pi$

$a\cdot b = |a||b| \cos\pi = -|a||b|$

$a \cdot b = |a||b|\underbrace{\cos\theta}_{\le 1} \le |a||b|$

$\frac{a \cdot b}{|a||b|} = \cos\theta$

$a \cdot b \neq 0$



Properties of inner product:

1) $a \cdot b = b \cdot a$

2) $a \cdot (\lambda b) = \lambda (a \cdot b)$ scalar property

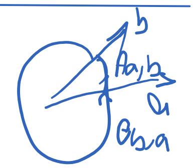
3) $a \cdot (b + c) = a \cdot b + a \cdot c$

distributive with respect to vector addition

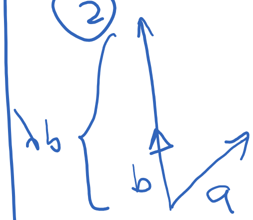
① $b \cdot a = |b||a|\cos\theta_{ba}$

$= |a||b|\cos(360° - \theta_{a,b})$

$= |a||b|\cos\theta_{a,b} = a \cdot b$



②



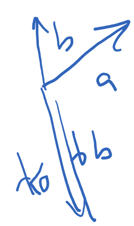
$a \cdot \lambda b = |a||\lambda b|\cos\theta_{a,\lambda b}$

$= |a|\lambda|b|\cos\theta_{a,b}$

show in each case

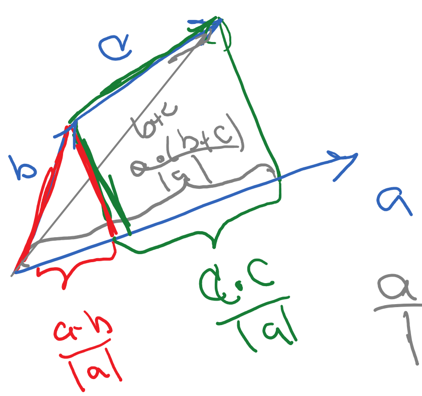
$= \lambda |a||b|\cos\theta_{a,b}$

$= \lambda\, a \cdot b$



$a \cdot (-\lambda b) = -\lambda(a \cdot b)$

③



$\frac{a \cdot (b+c)}{|a|} = \frac{a \cdot b}{|a|} + \frac{a \cdot c}{|a|}$

$\longrightarrow$ we get property 3

Note about HW1

5. (**20 Points**) Show that the components of the inverse of a matrix satisfy,

5. (**20 Points**) Show that the components of the inverse of a matrix satisfy,

$$A^{-1}_{rk} = \frac{1}{2 \det A} \epsilon_{ijk} \epsilon_{pqr} A_{ip} A_{jq} \tag{6}$$

Hint: Compute $A^{-1}_{rk} A_{km}$.

This is very important

$$= (A^{-1} A)_{rm} = \delta_{rm}$$