Before continuing with vector spaces I prove a few very important identities

Recall that we had

$$\det A = \frac{1}{6}\epsilon_{ijk}\epsilon_{mnp}A_{im}A_{jn}A_{kp}$$

a similar relation is having $\epsilon$'s on the LHS

$$\epsilon_{ijk}\epsilon_{mnp}\det A = \det \begin{array}{c|c} \begin{array}{ccc} \text{col } m & \text{col } n & \text{col } p \end{array} & \\ \begin{bmatrix} A_{im} & A_{in} & A_{ip} \\ A_{jm} & A_{jn} & A_{jp} \\ A_{km} & A_{kn} & A_{kp} \end{bmatrix} & \begin{array}{c} \text{row } i \\ \text{row } j \\ \text{row } k \end{array} \end{array}$$

Lets use $A = \delta$ identity matrix

$$\det \delta = \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1$$

$$\boxed{\begin{aligned} \epsilon_{ijk}\epsilon_{mnp} &= \det \begin{bmatrix} \delta_{im} & \delta_{in} & \delta_{ip} \\ \delta_{jm} & \delta_{jn} & \delta_{jp} \\ \delta_{km} & \delta_{kn} & \delta_{kp} \end{bmatrix} \\ &= \delta_{im}(\delta_{jn}\delta_{kp} - \delta_{kn}\delta_{jp}) - \delta_{in}(\delta_{jm}\delta_{kp} - \delta_{km}\delta_{jp}) \\ &\quad + \delta_{ip}(\delta_{jm}\delta_{kn} - \delta_{km}\delta_{jn}) \end{aligned}}$$

eq1

$\epsilon \longleftrightarrow \delta$

Let's sup $p = k$

$$\epsilon_{ijk}\epsilon_{mnk} = \delta_{im}(\delta_{jn}\overbrace{\delta_{kk}}^{3} - \overbrace{\delta_{kn}\delta_{jk}}^{\delta_{jn}}) - \delta_{in}(\delta_{jm}\overbrace{\delta_{kk}}^{3} - \overbrace{\delta_{km}\delta_{jk}}^{\delta_{mj}}) + \delta_{ik}(\delta_{jm}\delta_{kn} - \delta_{km}\delta_{jn})$$

$$= 3\delta_{im}\delta_{jn} - \delta_{im}\delta_{jn} - 2\delta_{in}\delta_{jm} + \underbrace{\delta_{ik}\delta_{kn}}_{\delta_{in}}\delta_{jm} - \underbrace{\delta_{ik}\delta_{km}}_{\delta_{im}}\delta_{jn}$$

$$= 2\delta_{im}\delta_{jn} - 2\delta_{in}\delta_{jm}$$

$+\delta_{in}\delta_{jm} \to \delta_{im}\delta_{jn} \quad = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$

$$\boxed{\epsilon_{ijk}\epsilon_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}}$$ eq2

Very important

---

Next one

$n = j \quad n \to j$

$$\epsilon_{ijk}\epsilon_{mjk} = \delta_{im}\underbrace{\delta_{jj}}_{3} - \underbrace{\delta_{ij}\delta_{jm}}_{\delta_{im}}$$

$$= 2\delta_{im}$$

$$\boxed{\epsilon_{ijk}\epsilon_{mjk} = 2\delta_{im}}$$ eq3

---

$m \to i$

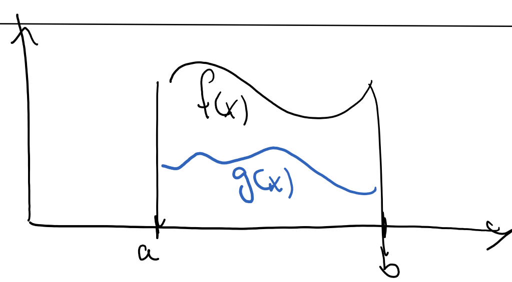
$$\epsilon_{ijk}\epsilon_{ijk} = 2\underbrace{\delta_{ii}}_{3} = 6$$

$$\boxed{\epsilon_{ijk}\epsilon_{ijk} = 6}$$ eq 4

---

Going back to vector space and inner product

Nontrivial examples of vector spaces




$\Omega = [a, b]$

$f+g$ is defined $\exists$ such that

then $(f+g)(x) := f(x) + g(x)$

---

Scalar product $\lambda \in \mathbb{R}$

$\lambda f$

is defined $\ni$

$\forall x \in \Omega \quad (\lambda f)(x) = \lambda (f(x))$

---

$\bar{V} = \{$ functions on $\Omega \to \mathbb{R} \}$ $\quad\quad 0$ defined as

$(\bar{V}, +, \circ)$ is a vector space $\quad\quad \forall x \in \Omega \quad \underbrace{0}_{\text{function}}(x) = \underbrace{0}_{\text{number zero}}$

---

What should we prove?

A1) $f + g = g + f$ commutative $\quad$ Addition

A2) $f + (g + h) = (f + g) + h$ associative

A3) $f + \underbrace{0}_{\text{function zero}} = f$ zero member

---

B1) $(\lambda \mu) f = \lambda (\mu f)$ scalar product property

B2) $(\lambda + \mu) f = \lambda f + \mu f$ scalar addition distribution

B3) $\lambda (f + g) = \lambda f + \lambda g$ vector " "

B4) $1 f = f$

---

All these properties should be proved. For example A1;

$f + g \stackrel{?}{=} g + f$

To prove this we need to show that this holds for any $x \in \Omega$

$\forall x \in \Omega \quad (f + g)(x) = (g + f)(x) \quad ?$

$(f + g)(x) = f(x) + g(x)$ commutative property of real numbers

$= g(x) + f(x)$

$\because = (g + f)(x) \quad \blacksquare$

other 6 properties are proved similarly.

# Subspace

Let's say $V$ is a vector space

$(V, \overset{u+v}{+}, \overset{\lambda u}{\cdot}, \overset{\lambda \in}{\mathbb{R}})$

We call $W \subseteq V$ a subspace if

$W$ is also a vector space.

$\forall u, v, w \in V$ → $\forall u, v, w \in W$

A1 $u+v = v+u$

A2 $u+(v+w) = (u+v)+w$

~~A3~~ $u+0 = 0+u$

B1 $(\lambda\mu)u = \lambda(\mu u)$

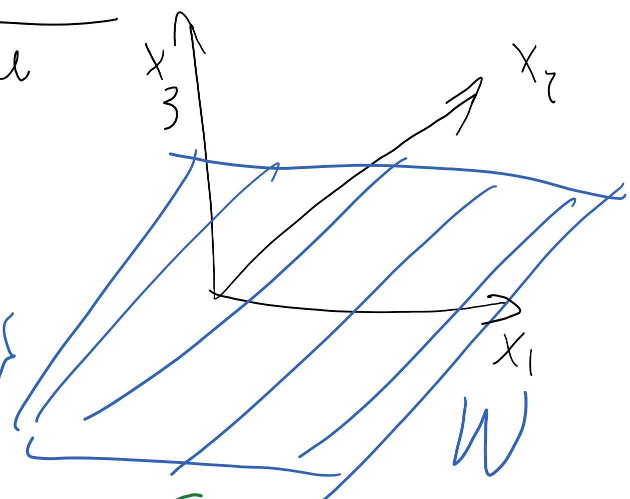
B2 ~~B3~~ $(\lambda+\mu)u = \lambda u + \mu u$

B4 $\lambda(u+v) = \lambda u + \lambda v$

$1u = u$

subspace

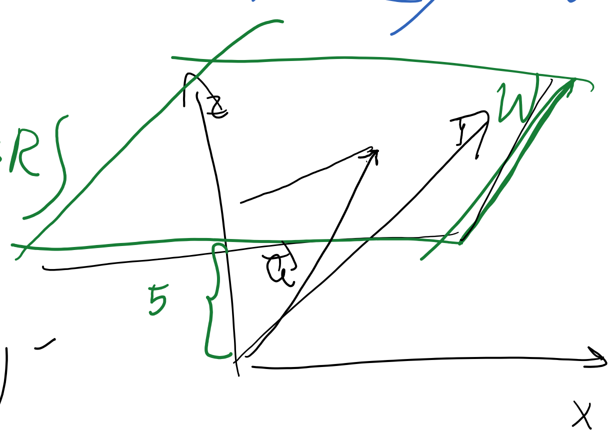
$V = \mathbb{R}^3, \overbrace{+, \cdot}^{\text{normal vector addition, ...}}$




Example 1 $W = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$

$W \subseteq V$

$W' = \{(x, y, 5) \mid x, y \in \mathbb{R}\}$




$a = (x_a, y_a, 5)$

$\lambda = 6$

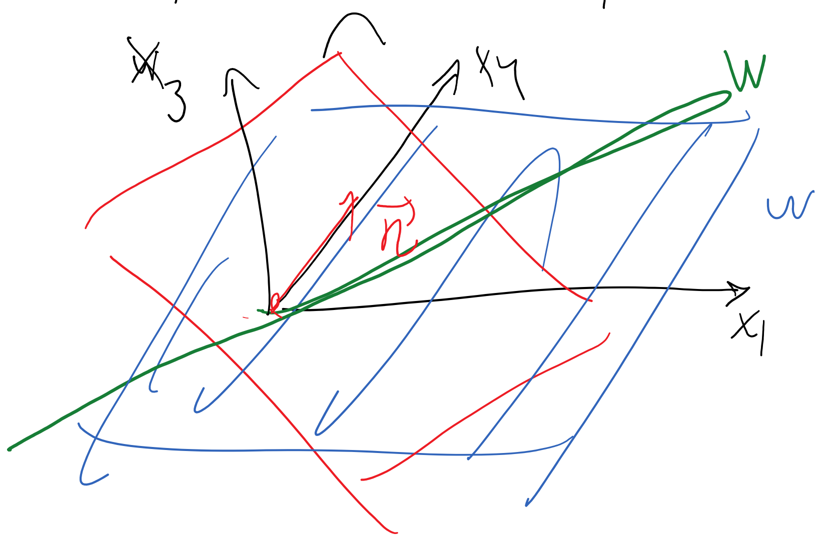
$\lambda a = (6x_a, 6y_a, 30) \notin W'$

$a + b = (x_a + x_b, y_a + y_b, 10) \notin W$

Geometric interpretation:

SUBSPACE is a FLAT

hyperplane that passes through zero




A subset W of a vector space V is a vector space with the same addition and scalar product if we only show that it's closed w.r.t. vector addition and scalar product.

$(V, +, \cdot, \mathbb{R})$ vector space

$W \subseteq V$

$\forall u, v \in W \quad u + v \in W \quad \Big\} \longrightarrow (W, +, \cdot, \mathbb{R})$ is a vector space

Interpretation: vector spaces are hyperplanes passing through zero.

shorter form $\left.\begin{array}{l} u, v \in W \\ \lambda \in W \end{array}\right\} \to u + \lambda v \in W$

Inner product vector spaces:

V is an inner product vector space if it's a vector space and is equipped with an inner product . that satisfies:

$u, v \in V$ $\quad u \circ v$ or $\langle u, v \rangle$ $\quad (u, u)$ is a real number satisfying

$a, b \in V$, $\lambda \in \mathbb{R}$

don't need to show this

1) $a \circ (\lambda b) = (\lambda a) \cdot b = \lambda (a \cdot b)$ $\qquad a \cdot (\lambda b) = \lambda (a \cdot b)$ scalar product homogeneity

2) $a \cdot (b + c) = a \cdot b + a \cdot c$ $\qquad$ dist. w.r.t. vector addition

3) $a \cdot b = b \cdot a$ $\qquad$ Commutativity

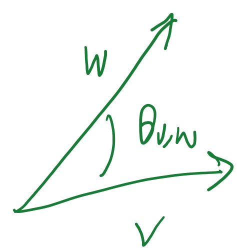
4) $a \circ a \geq 0$ & $a \circ a = 0 \iff a = 0$ $\qquad$ positive definite property

---

Recall for vectors in $\mathbb{R}^2, \mathbb{R}^3$ we had defined

$$\vec{v} \cdot \vec{w} = |v||w| \cos \theta_{v,w}$$

and showed it satisfied prop. 1 to 4 above.




$$|v \cdot w| \leq |v||w|$$

$$|v \cdot w| = |v||w| \, |\underbrace{\cos \theta_{u,w}}_{\leq 1}| \leq |v||w|$$

for $\mathbb{R}^2, \mathbb{R}^3$ vectors

BUT we can prove this inequality in general

---

For any inner product vector space we have the following inequality (called Cauchy Schwarz inequality)

$$|u \circ v| \leq |u||v| \qquad \text{CS}$$

Recall $|u| = \sqrt{u \circ u}$
$|v| = \sqrt{v \, v}$

Proof: $u, v$ are arbitrary &
$\alpha$ is any chosen real number

$$(u + \alpha v) \circ (u + \alpha v) \geq 0 \quad (4)$$

1) $a \circ (\lambda b) = (\lambda a) \cdot b = \lambda (a \cdot b)$

$$(u+\alpha v)\cdot(u+\alpha v) \geq 0 \quad (4)$$

$$(u+\alpha v)\cdot v + (u+\alpha v)\cdot(\alpha v) \quad (2)$$

$$v\cdot(u+\alpha v) + (\alpha v)\cdot(u+\alpha v) \quad (3)$$

$$u\cdot u + u\cdot(\alpha v) + (\alpha v)\cdot u + (\alpha v)\cdot(\alpha v) \quad (2)$$

$$|u|^2 + \alpha\, u\cdot v \qquad \alpha(u\cdot v) \qquad \alpha(\alpha v)\cdot v,\ \alpha(v\cdot\alpha v),\ \alpha\alpha(v\cdot v),\ \alpha^2|v|^2$$

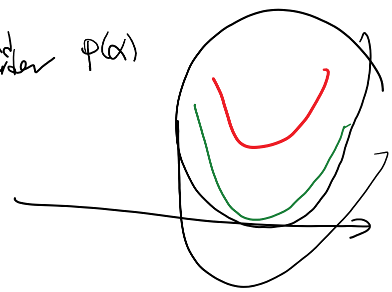
1) $a\cdot(\lambda b) = (\lambda a)\cdot b = \lambda(a\cdot b)$

2) $a\cdot(b+c) = a\cdot b + a\cdot c$

3) $a\cdot b = b\cdot a$

4) $a\cdot a \geq 0$ & $a\cdot a = 0 \Longleftarrow a=0$

2nd order $p(\alpha)$



$\forall \alpha \in \mathbb{R}$

$$\underbrace{|v|^2}_{A}\alpha^2 + \underbrace{2\,u\cdot v}_{B}\,\alpha + \underbrace{|u|^2}_{C} \geq 0$$

$$A\alpha^2 + B\alpha + C \geq 0$$

What can we say about $A, B, C$

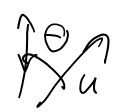
$$\Delta = B^2 - 4AC \leq 0$$

$$4(u\cdot v)^2 - 4|v|^2|u|^2 \leq 0$$

$$(u\cdot v)^2 \leq (|u||v|)^2 \qquad \text{take the square root}$$

$$\boxed{|u\cdot v| \leq |u||v|}$$

vectors $u, v$ can we define an angle between $u, v$? for vectors $u\cdot v = |u||v|\cos\theta_{u,v}$



$$\boxed{\cos\theta_{u,v} = \frac{u\cdot v}{|u||v|} \qquad u,v \neq 0}$$

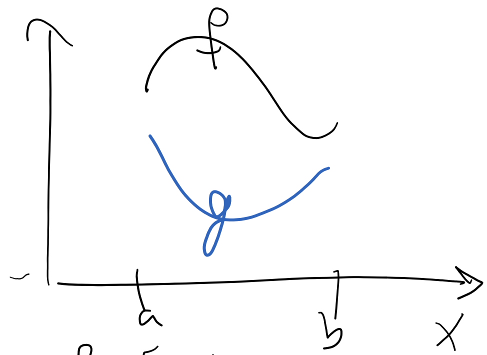
For any vector space

$$-1 \leq \frac{|u\cdot v|}{|u||v|} \leq 1$$

because of CS

Inner product between functions

$$f \cdot g \text{ or } \langle f, g \rangle = \int_{\Omega} f(x)\, g(x)\, dx$$




$\Omega = [a, b]$

in this case $\int_a^b f(x) g(x)\, dx$

Can I define angle between f, g ?!

yes

$$|f| = \sqrt{f \cdot f} = \sqrt{\int_a^b f^2\, dx}$$

$$\cos\theta_{f,g} = \frac{f \cdot g}{|f|\,|g|}$$

In inner product vector spaces we can talk about magnitude |f|
and angles $\theta_{f,g}$