CM2021/09/20

Thursday, September 2, 2021 4:31 PM

Why the definition we provided is an inner-product?

$f\cdot(\lambda g) = \int_\Omega f(x)\cdot(\lambda g)(x)\,dx = \int f(x)(\lambda g(x))dx = \lambda\int_\Omega f(x)g(x)dx = \lambda(f\cdot g)$ (1) ✓

1) $a\cdot(\lambda b) = (\lambda a)\cdot b = \lambda(a\cdot b)$

2) $a\cdot(b+c) = a\cdot b + a\cdot c$

3) $a\cdot b = b\cdot a$

4) $a\cdot a \geq 0$ & $a\cdot a = 0 \Longleftarrow a=0$

$a = \sqrt{a\cdot a}$

$f\cdot(g+h) = \int f(x)\cdot\underbrace{(g+h)(x)}_{(g(x)+h(x))} = \ldots = f\cdot g + f\cdot h$

$f\cdot g = \int f(x)g(x)dx = \int g(x)f(x)dx = g\cdot h$

$f\cdot f = \int_\Omega \underbrace{f(x)f(x)}_{|f(x)|^2\geq 0}dx \geq 0$ $\quad f\cdot f = 0$ iff $f(x)=0$ $\forall x\in\Omega$, $\underline{f=0}$

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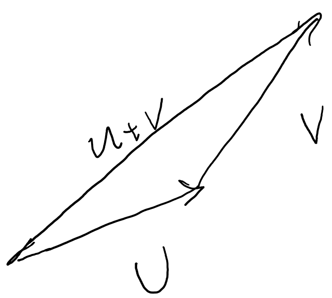
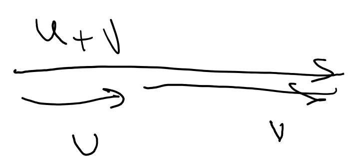
The definition above is an inner product. I just missed some crucial point that I'll get to it soon.




$\underline{|u+v| \leq |u|+|v|}$

$=$ if $u$ is aligned with $v$




Recall

$\underline{\text{if we have}}$ inner product $\rightarrow |u| = \sqrt{u\cdot u}$

In fact, for any inner product space, we have the triangular inequality. We can prove it by:

Show $|u+v|^2 = (u+v)\cdot(u+v) \leq (|u|+|v|)^2$ you'll use CS

$\|u+v\|^2 = (u+v)\cdot(u+v) \leq (\|u\|+\|v\|)^2$ you'll use CS

$\sqrt{\phantom{x}}$ $\Downarrow$

Tri_equality

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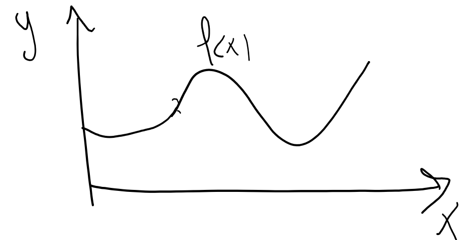
$$V = \{ f \mid f \text{ function } \Omega = (a,b) \to \mathbb{R} \}$$




$(V, +, \circ, \mathbb{R})$ vector space ☺

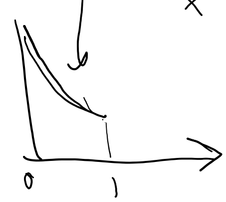
- We cannot define inner product -> we cannot talk about length and angle

$(V, V) \to \mathbb{R}$

$(a,b) = (0,1)$

$$(f, g) = \int_0^1 f(x)\,g(x)\,dx < \infty$$

$f(x) = \frac{1}{x}$




$$f\cdot f = \int_0^1 \frac{1}{x^2}\,dx = \frac{-1}{x}\Big|_0^1 = \infty$$ ☹

- For general functions we cannot define inner product, because f.f or f.g 's cannot always be computed or are not finite!

How about we define the SUBSET W of V for which we have the following property:

$$\underline{W} = \{ f \in V \mid \underbrace{f\cdot f = \int_{\Omega} f(x) f(x)}_{\|f\|^2 < \infty \quad \|f\| < \infty} < \infty \}$$

$\frac{1}{x} \notin W$

1) Can we define Inner product for functions in W?

$f \in W,\ g \in W$ Does $f\cdot g$ exist & $|f\cdot g| < \infty$

$$|f\cdot g| = \int_{\Omega} f(x)\,g(x)\,dx < \infty$$

CS $\underbrace{|f.g|}$ $\leq \underset{<\infty}{|f|}\underset{<\infty}{|g|}$

f.g can be calculated & $<\infty$

We can define inner product in W
This may be an inner product vector space

1) $f \in W, g \in W$ $f+g \in W$: $|f+g| < \infty$ why?
2) $f \in W, \lambda \in R$ $\lambda f \in W$

$f \in W: |f| < \infty$
$g \in W \ |g| < \infty$
$|f+g| < |f| + |g|$ Tri. En

$\rightarrow |f+g| < \infty \rightarrow f+g \in W$

$|\lambda f| = |\lambda||f|$ $< \infty \rightarrow \lambda f \in W$

$|\lambda|$: real #

$|f| < \infty$ because $f \in W$

So W is a subspace of V. It's basically a subspace of functions that is equipped with an inner product. so for these functions we can talk about their norm (magnitude) and angle between functions (concepts such as f is normal to g)

$f: \mathbb{R}^n \rightarrow \mathbb{R}$

$$\underbrace{L^2 \Omega}_{\downarrow} = \left\{ f \,\middle|\, \begin{array}{l} \Omega \rightarrow R \\ \Omega \subseteq \mathbb{R}^n \end{array} \Rightarrow \begin{array}{l} \int_\Omega f.f \, dv < \infty \\ |f|_\Omega < \infty \end{array} \right\}$$

this is an inner product space.

Summary: L2 space of functions is a very nice subset of functions for which we can define inner product -> (length, angle)

We are not always fortunate enough to work with L2 functions in practice. There are functions (or other members of vector spaces) that don't have an inner product.

So, what is the next best thing?

| A1-3, B1-4 | Normed VS | IP VS |
| --- | --- | --- |
| Vector space | V.S + norm | V.S. + inner product |
| - length ✗<br>- angle ✗ | $\|v\|$, $\|u\|$ ✓, $\theta$ ✗<br>- length | $\|v\|$, $\theta_{uv}$<br>- length ✓<br>- angle ✓ |

Stronger →

(•)
1) $f \cdot \lambda g = \lambda f \cdot g$
2) $f \cdot (g+h) = f \cdot g + f \cdot h$
3) $f \cdot g = g \cdot h$
4) $f \cdot f \geq 0$ $\quad f \cdot f = 0 \equiv f = 0$

$|f| = \sqrt{f \cdot f}$

(B)

IP_based norm

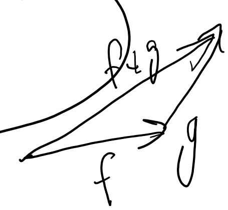
$\longrightarrow \quad f := \sqrt{f \cdot f}$

B1 $|\lambda f| = |\lambda| |f|$

B2 $|f| \geq 0$ & $|f| = 0$ iff $f = 0$

B3 $|f+g| \leq |f| + |g|$

$\sqrt{(\lambda f) \cdot (\lambda f)} = \sqrt{\lambda (\lambda f \cdot f)} = \sqrt{\lambda^2 f \cdot f} = |\lambda| \sqrt{f \cdot f} = |\lambda| |f|$




In fact any function from V -> non-negative real numbers that has properties 1 to 3 is called a **normed vector space**. Normed vector space can only talk about length (norm) not angles!

$\| \quad \| \qquad | \quad |$

(B1) $\|\lambda f\| = |\lambda| \, \|f\|$

(B2) $\|f\| \geq 0 \qquad \|f\| = 0$ iff $f = 0$

(B1) $\|\lambda f\| = |\lambda| \, \|f\|$

(B2) $\|f\| \geq 0 \qquad \|f\| = 0 \text{ iff } f = 0$

(B3) $\|f+g\| \leq \|f\| + \|g\|$

$\|\cdot\|$ is a norm

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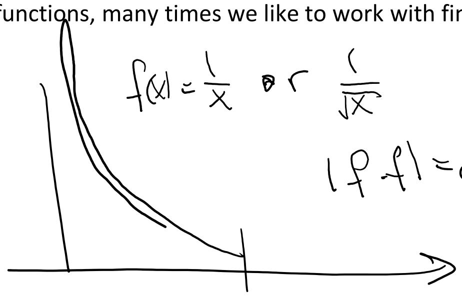
Are there any practical examples of normed spaces that are not inner product spaces? YES

For functions, many times we like to work with finite functions (that the function does not blow up in the set considered)

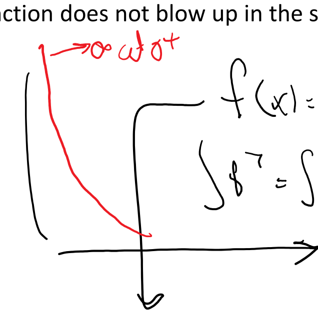


$f(x) = \frac{1}{x}$ or $\frac{1}{\sqrt{x}}$

$|f \cdot f| = \infty$

$\int_0^1 f \cdot f \, dx = \infty$

$\Omega = (0, 1]$



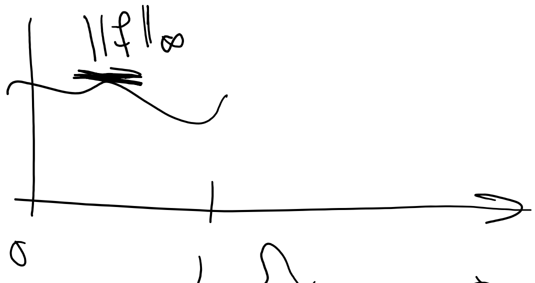

$f(x) = \frac{1}{x^{1/4}}$

$\int f^2 = \int_0^1 x^{-\frac{1}{2}} = \frac{1}{\frac{1}{2}} \sqrt{x} \Big|_0^1 < \infty$

$L^2(\Omega)$

$(\cdot)$ is good for this

Maybe we don't like 1/x^0.25 because it blows up at 0. So, how about finite functions?




$f_M = \max_{x \in \Omega} f(x) < \infty$

$\|f\|_\infty$

$L^\infty_\Omega = \{ f \mid \|f\|_\infty < \infty \}$

$\frac{1}{x^{1/4}} \notin L^\infty_\Omega$

Talking about finite functions || . ||$_\infty$ is a natural norm, but we don't have an inner product and no angle!

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Side note: Let's try to be too smart and define an inner-product from a norm?

$(\cdot) \longrightarrow \|\,\|$ $\qquad \|a\| = \sqrt{a \cdot a}$ it is a norm R1 ✓ R2 ✓ R3 ✓

$\|\cdot\| \xrightarrow{?} (\cdot)$ if we had an inner product

$$\frac{\|a+b\|^2 - \|a-b\|^2}{4} = a \cdot b$$

$$\|\cdot\| \longrightarrow (\cdot)$$

But unfortunately the . We define this way from a norm, does not satisfy all 4 conditions of an inner product:(

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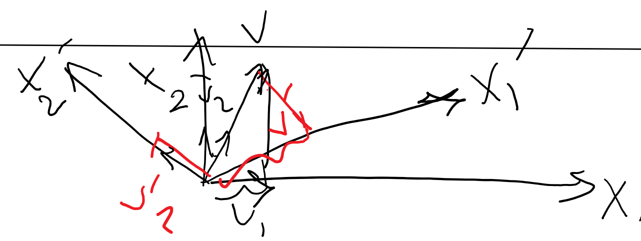
Coordinates and coordinate transformation:




$(v_1', v_2')$ $\qquad (v_1, v_2)$

All these discussions below are for a general vector space (with maybe some minor tweaks for functions) because they are all built on vector space (plus possibly inner product) concept(s)

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Linear Independence:

$v_1, v_2, v_3, \ldots, v_q \in V$ are called linearly independent if

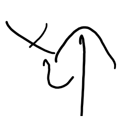
vector space

for $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ $\qquad \alpha_1 v_1 + \alpha_2 v_2 \cdots = 0 \longrightarrow \alpha_i = 0$ MUST be zero

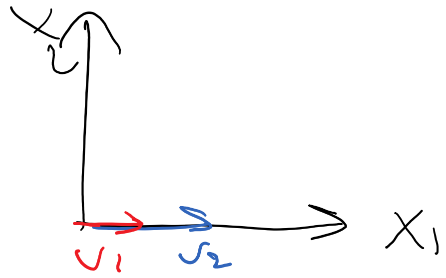
opposite of this (at least 1 $\alpha$, say $\alpha_j$, is nonzero $\rightarrow$

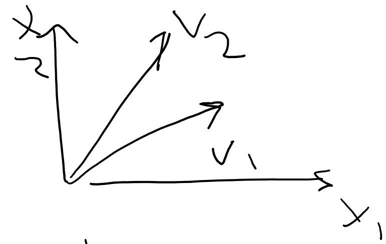
$/\alpha_j$

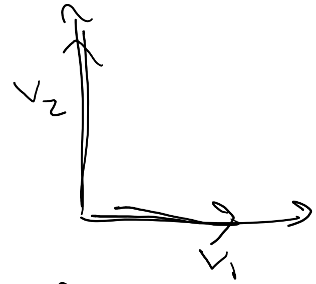
$$\underbrace{v_j = \beta_1 v_1 \cdots + \beta_{j-1} v_{j-1} + \beta_{j+1} v_{j+1} \cdots}_{\text{linearly dependent}}$$



linearly dependent



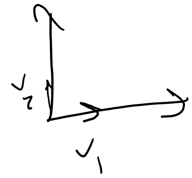



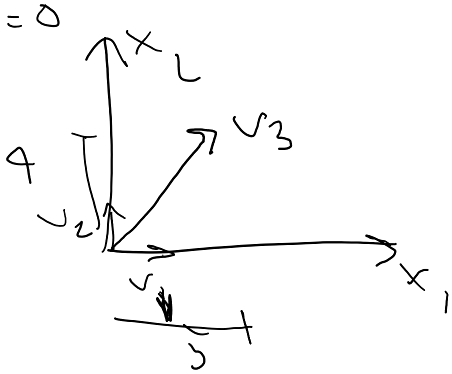

$v_2 = 2v_1$

linearly dep.

linearly independent




$2v_1 - v_2 = 0$
(2) (-1)




$V_3 = 5V_1 + 4V_2$

Not linearly indep.

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Basis for a vector space:

$e = \{e_1, e_2, \cdots, e_n\}$ is called a basis for a vector space, if

(1) $e_1, \ldots, e_n$ are linearly independent

(2) Any vector $v \in D$ can be expressed in terms of the basis functions

$v \in D$

$$v = v_1 e_1 + v_2 e_2 + v_3 e_3 + \cdots + v_n e_n$$

$(v_1, \ldots, v_n)$ is called the coordinate of $v$

w.r.t $(e_1, e_2, \ldots, e_n)$

n is the dimension of the vector space

Coordinate of $v$ w.r.t $e$ is unique

$$V = v_1 e_1 + \dots + v_n e_n$$
$$v_1' e_1 + \dots + v_n' e_n$$

$$\underbrace{0 = (v_1 - v_1')e_1 + \dots + (v_n - v_n')e_n}$$ VS properties

$e_1 \dots e_n$ are linearly indep. $\rightarrow$ $v_i - v_i' = 0$ $\rightarrow$

$$(v_1, \dots, v_n) = (v_1', \dots, v_n')$$

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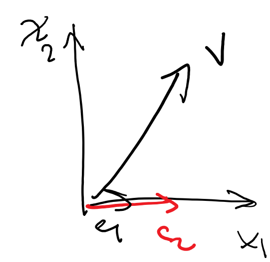
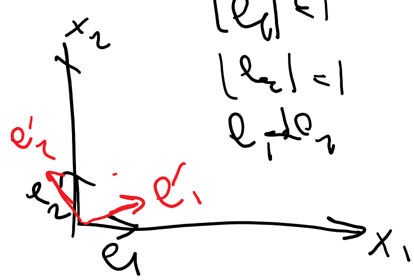
Examples 2D space




① & ② are violated

not a basis




$|e_1| = 1$

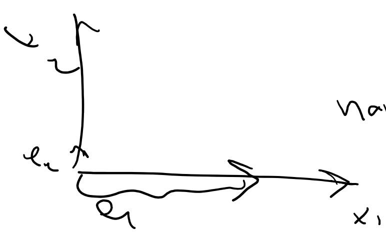
$|e_2| = 1$

$e_1 \perp e_2$

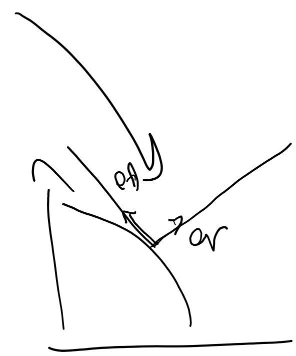
$e_i \cdot e_j = \delta_{ij}$

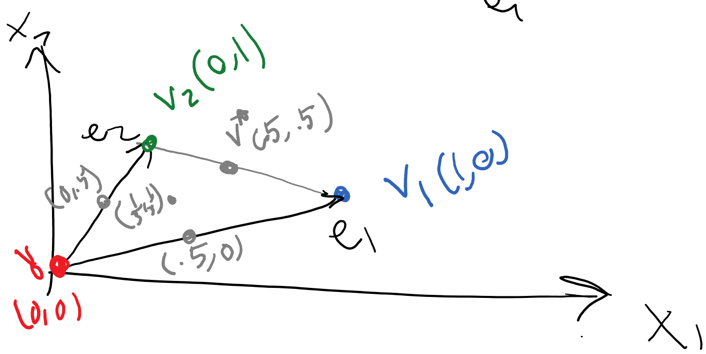
orthonormal basis

$e_i' \cdot e_j' = \delta_{ij}$




normal basis




this is a basis

$V_0 = (0,0)$

$V_0 = 0e_1 + 0e_2$

$V_1 : (1,0)$

$V_1 = (1)e_1 + 0e_2 = e_1$

$V_2 : (0,1)$

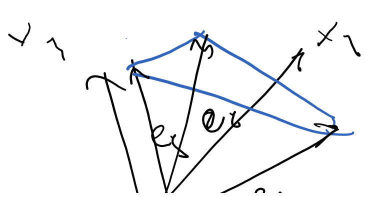
$V_2 = 0e_1 + 1e_2 = e_2$

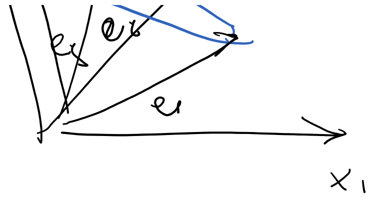
Natural coordinate system for a triangle

The values above are natural coordinates of points in a triangle

$v^*$ $(.5, .5)$

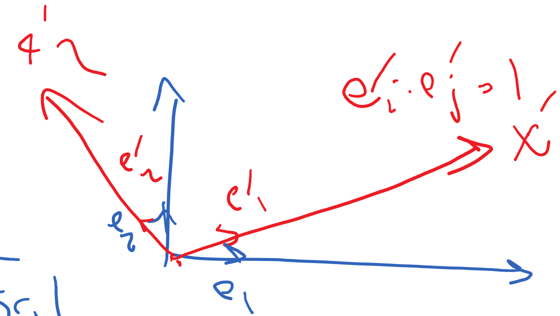
$v^* = .5e_1 + .5e_2$






Orthonormal basis: that individual basis vectors of size 1 and they are normal

Obviously this definition ONLY makes sense in an inner product vector space




$e_1 \cdot e_1 = 1$

$e_1 \cdot e_2 = 0$

$e_2 \cdot e_2 = 1$

$\boxed{e_i \cdot e_j = \delta_{ij}}$

meaning of $V_i$ in orthormal basis

$V_i' \longleftrightarrow V_i'$

$T_{ij}' \longleftrightarrow T_{ij}'$

$e_i' \longleftrightarrow e_i$

$L_2^{\Omega}$

What is the dimension of $L_2^{\Omega}$

$\infty$