CM2021/09/07

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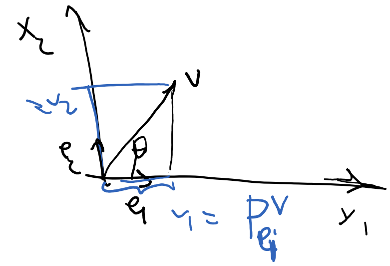
Continue: Orthonormal basis in an inner product vector space:

$e_1 \cdot e_1 = 1 \quad e_1 \cdot e_2 = 0$

$e_2 \cdot e_1 = 0 \quad e_2 \cdot e_2 = 1$

$e_i \cdot e_j = \delta_{ij}$



Interpretation of components of a vector when expressed in an orthonormal basis

$V = V_i e_i \qquad \underline{V_i = ?} \qquad$ orthonormal basis

$\underline{V \cdot e_j} = (V_i e_i) \cdot e_j = V_i (\underbrace{e_i \cdot e_j}_{\delta_{ij}}) = V_i \delta_{ij} = V_j$

meaning of $V \cdot e_j$

$V \cdot e_j = |V| \cos\theta_{V e_j} \underbrace{|e_j|}_{1} = \underbrace{|V| \cos\theta_{V,e_j}} = P^V_{e_j}$

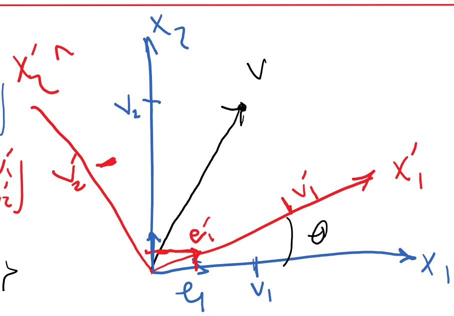
$$V = V_i e_i \Rightarrow V_i = V \cdot e_i = P^V_{e_i} \quad \text{for an orthonormal basis}$$

Coordinate transformation

$V = V_i e_i = V_1 e_1 + V_2 e_2 \qquad [V]_e = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$

$V = V_i' e_i' = V_1' e_1' + V_2' e_2' \qquad [V]_{e'} = \begin{bmatrix} V_1' \\ V_2' \end{bmatrix}$

How are $[V]_e$ & $[V]_{e'}$ are related?



The starting point is expressing the components of one basis set in the other basis set.

In general (2D, 3D) we write $\quad e_i' = Q_{ij} e_j$.

For this 2D example $\quad e_1' = \cos\theta\, e_1 + \sin\theta\, e_2$

$\qquad e_2' = -\sin\theta\, e_1 + \cos\theta\, e_2$

$$Q = \begin{bmatrix} e_1' \\ \hline e_2' \end{bmatrix}_{\text{expressed in } e \text{ system}} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$e_i' \cdot e_k = Q_{ik} \qquad$ It should be

$e_i' \cdot e_k = (Q_{ij} e_j) \cdot e_k = Q_{ij} \overbrace{e_j \cdot e_k}^{\delta_{jk}} = Q_{ij} \delta_{jk} = Q_{ik}$

Summary

$e'_i = Q_{ij} e_j \longrightarrow Q_{ij} = e'_i \cdot e_j$

$e_j = Q_{ij} e'_i$ — Components of $\{e'\}$ are expressed in $\{e\}$ coordinate system

in 3D

$$Q = \begin{bmatrix} e'_1 \\ \hline e'_2 \\ \hline e'_3 \end{bmatrix}$$

→ unit vectors of ( )′ expressed in ( ) system

In TAM551 λ is used instead of Q

Before coordinate transformation, we show Q is an orthogonal matrix.

$e'_i = Q_{ij} e_j \longleftrightarrow Q_{ij} = e'_i \cdot e_j$

inverse relation: $e_j = R_{ji} e'_i \longrightarrow R_{ji} = e_j \cdot e'_i = Q_{ij} = Q^T_{ji}$

seek R?

$\longrightarrow R = Q^{-1} = Q^T$ definition of orthogonal matrix

$\{e'\} = Q\{e\} \longrightarrow$

$\{e\} = Q^{-1}\{e'\}$ ($Q^{-1} = R$)

$(Q^T Q)_{ij} = Q^T_{ik} Q_{kj} = Q_{ki} Q_{kj} = \delta_{ij}$

$(Q Q^T)_{ij} = Q_{ik} Q^T_{kj} = Q_{ik} Q_{jk} = \delta_{ij}$

orthogonal matrix

$$Q^T Q = Q Q^T = I$$

($Q_{ki} Q_{kj} = \delta_{ij}$, $Q_{ik} Q_{jk} = \delta_{ij}$)

(basically $Q^T = Q^{-1}$)

Interpretation of why $QQ^T = I$:

$$QQ^T = \begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix} \begin{bmatrix} e'_1 \\ \hline e'_2 \\ \hline e'_3 \end{bmatrix}_{3\times 3} \overset{\begin{matrix} c_1 & c_2 & c_3 \end{matrix}}{\begin{bmatrix} e'_1 & e'_2 & e'_3 \end{bmatrix}}_{3\times 3} = \begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix} \overset{\begin{matrix} c_1 & c_2 & c_3 \end{matrix}}{\begin{bmatrix} e'_1 \cdot e'_1 & e'_1 \cdot e'_2 & e'_1 \cdot e'_3 \\ e'_2 \cdot e'_1 & e'_2 \cdot e'_2 & e'_2 \cdot e'_3 \\ e'_3 \cdot e'_1 & e'_3 \cdot e'_2 & e'_3 \cdot e'_3 \end{bmatrix}}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

why? $e'_i \cdot e'_j = \delta_{ij}$

$$= \begin{bmatrix} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

why? $e'_i \cdot e'_j = \delta_{ij}$

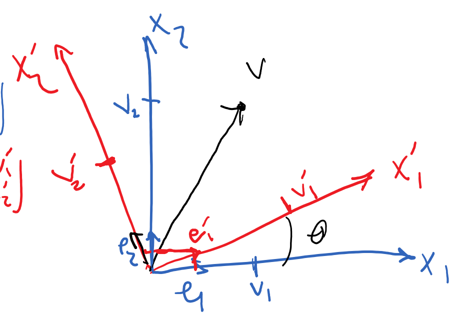
Interpretation: Any orthogonal matrix is a matrix whose rows are orthonormal and at the same time its columns are orthonormal. Such matrix can represent a coordinate transformation and a coordinate transformation matrix Q also is also orthonormal.

Orthonormal bases <-> Q is orthogonal

What is the real use of Q?

$V = V_i e_i = V_1 e_1 + V_2 e_2 \quad [V]_e = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$

$V = V'_i e'_i = V'_1 e'_1 + V'_2 e'_2 \quad [V]_{e'} = \begin{bmatrix} V'_1 \\ V'_2 \end{bmatrix}$



$V = V_j e_j$
$e_j = Q_{ij} e'_i$
$\rightarrow V = V_j (Q_{ij} e'_i) = (Q_{ij} V_j) e'_i = V'_i e'_i$

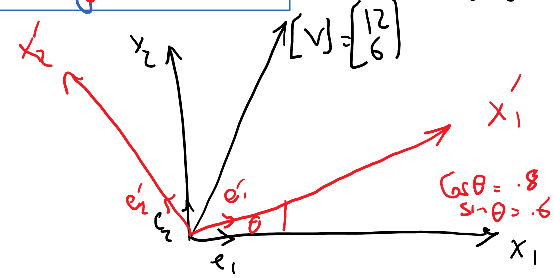
$$V'_i = Q_{ij} V_j \qquad [V]' = [Q][V]$$

$$V_j = Q_{ij} V'_i \qquad [V] = [Q]^T [V]'$$

$V_j = Q^T_{ji} V'_i$

$Q = \begin{bmatrix} e'_1 \\ e'_2 \end{bmatrix} = \begin{bmatrix} .8 & .6 \\ -.6 & .8 \end{bmatrix}$

$[V] = \begin{bmatrix} 12 \\ 6 \end{bmatrix}$



$$[V]' = [Q][V] = \begin{bmatrix} .8 & .6 \\ -.6 & .8 \end{bmatrix} \begin{bmatrix} 12 \\ 6 \end{bmatrix} = \begin{bmatrix} .96 + .36 \\ -.72 + .48 \end{bmatrix} = \begin{bmatrix} 1.32 \\ -.24 \end{bmatrix}$$

tensor order

0 scalar $\phi' = \phi$ no Q (example temperature, speed, ... diffusion concentration

1	vector	$v'_i = Q_{im} v_m$	speed ... diffusion concentration; disp, velocity, acceleration
2	2nd order tensor	$T'_{ij} = Q_{im} Q_{jn} T_{mn}$	stress & strain
	4th order tensor	$C'_{ijkl} = Q_{im} Q_{jn} Q_{kp} Q_{lq} C_{mnpq}$	elasticity tensor
	any order mth	$T'_{i_0 i_1 i_2 \dots i_m} = Q_{i_0 j_0} Q_{i_1 j_1} \dots Q_{i_m j_m} T_{j_0 j_1 \dots j_m}$	

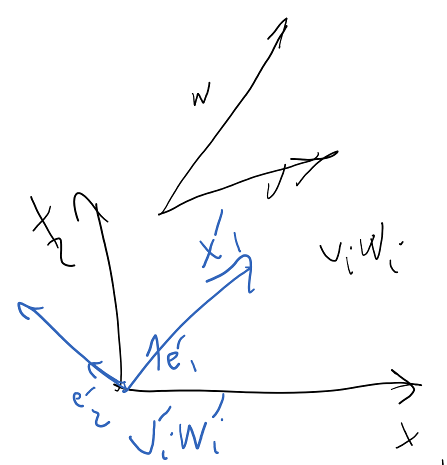
Speed = mag. Of vel (vec)

$v \cdot w = |v| |w| \cos\theta$

Q? Is magnitude of a vector a scalar?

$v = \sqrt{v \cdot v}$

I will show you for an orthonormal basis



(1) (✪) $v \cdot w = v_i w_i$

imagine if (✪) was the definition?

Then we must prove that it's actually a scalar, that is any coordinate system that is used, the same value is computed. (2)

$$v \cdot w = (v_i e_i)(w_j e_j) = v_i w_j \, e_i \cdot e_j$$

$$= \begin{cases} v_i w_j \delta_{ij} = v_i w_i \checkmark & \text{case 1 orthonormal coordinate system } e_i \cdot e_j = \delta_{ij} \\ v_i w_j g_{ij} & \text{case 2 general coordinate system } e_i \cdot e_j \neq \delta_{ij} \end{cases}$$

$e_i \cdot e_j = g_{ij}$ metric

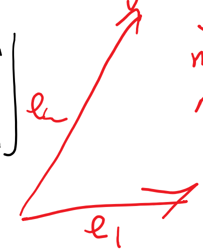
$$v \cdot w = \begin{cases} v_i \cdot w_i & \text{orthonormal basis} = [v_1 \dots v_n] \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \end{cases}$$

$$v \cdot w = \begin{cases} v_i w_i & \\ v_i\, g_{ij}\, w_j & \text{general basis} \quad g = e_i \cdot e_j \end{cases}$$

$$= [v_1 \cdots v_n] \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$$

$$= [v_1 \cdots v_n] \begin{bmatrix} g_{11} & \cdots & g_{1n} \\ & & \\ g_{n1} & \cdots & g_{nn} \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$$

$e_i \cdot e_j = g_{ij}$ metric matrix



Part 4

Now, let's go back to orthonormal bases, and show that vi.wi is in fact a scalar.

$v_i w_i \longrightarrow$ want to form $v'_m w'_m$

$v_i = Q_{mi} v'_m$

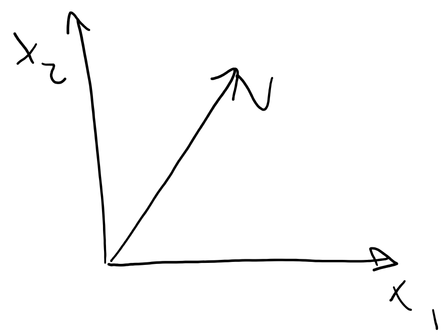
$w_i = Q_{ni} v'_n$

$$v_i w_i = \underbrace{Q_{mi} Q_{ni}}_{Q_{mi} Q^T_{in} = (QQ^T)_{mn} = \delta_{mn}} v'_m w'_n$$

$$= \delta_{mn} v'_m w'_n = v'_m w'_m$$

So in fact, even if we had taken a bad route of defining v.w based on vector components (vi.wi) for orthonormal basis, the definition would have been a scalar.

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$



$\star \; l_1(v) = |v_1| + |v_2|$

$l_2(v) = \sqrt{|v_1|^2 + |v_2|^2} = \sqrt{v \cdot v} \longrightarrow$ scalar

$l_p(v) = \sqrt[p]{|v_1|^p + |v_2|^p}$

$\star \; l_\infty(v) = \lim_{p \to \infty} l_p(v) = \max |v_i|$