CM2021/09/09

Thursday, September 9, 2021 4:31 PM

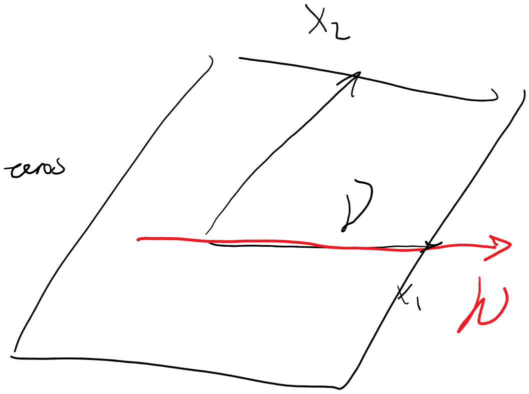
Linear Operators <-> second order tensors

A linear operator L, is a function from vector space V to vector space W that satisfies the following two properties:

$L : V , W$

both are vector spaces

0_V 0_W are there zeros



(P1) $\forall v_1, v_2 \in V$ $\overbrace{L(v_1+v_2)}^{\in W} = \overbrace{L(v_1)}^{\in W} + \overbrace{L(v_2)}^{W}$

(P2) $\forall v_1 \in V, \lambda \in R$ $L(\lambda v_1) = \lambda L(v_1)$

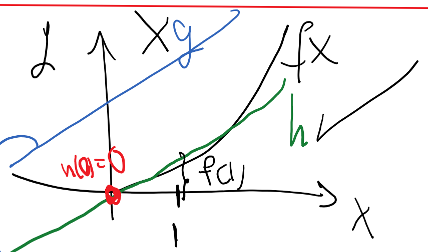
or alternatively just need to show $L(v_1+\alpha v_2) = L(v_1) + \alpha L(v_2)$

$V = R$ $W = R$

if f is linear

$$f(x) = f(\overset{\in R}{x} \cdot 1) = x \underbrace{f(1)}_{c} = xc$$

P2



In

For any linear operator L, $\boxed{L(0_V) = 0_W}$

$$= L(\overbrace{0_V + 0_V}^{0_V}) = \cancel{L(0_V)} + L(0_V)$$

$$L(0_V) =$$

$$0_W + \cancel{L(0_V)} =$$

$$\boxed{L(0_V) = 0_W}$$

Side note

Affine $y = cx + b$

matrix form

Affine $\underset{\sim}{y} = Ax \cancel{+} b$

vector vector

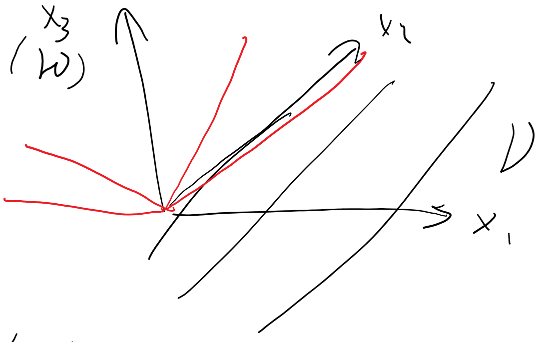
useful $Ax = b \equiv y = 0$

Example 2

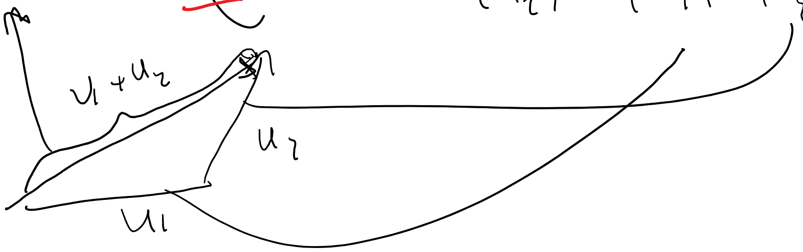
$L(u) = |u|$

$V = R^2, \; W = R$

$R^2 \to R$



$$L(u_1 + u_2) = |u_1 + u_2| \overset{?}{\leq} L(u_1) + L(u_2) = |u_1| + |u_2|$$



So this clearly is not a linear operator

$$L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \overbrace{\underbrace{\begin{bmatrix} 5 & 7 \\ 2 & 6 \\ 11 & -5 \end{bmatrix}}_{3\times 2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}^{\in W = R^3}$$

linear $\qquad R^2 \to R^3$

$\vec{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in V = R^2$

$R^2 \to R^3$

$V = \{ \text{functions } \mathbb{R} \to \mathbb{R} \}$

$V_0 = \{ f: \mathbb{R} \to \mathbb{R}$ are C^1 $(\overset{f}{\text{value}}, \overset{f'}{\text{derivative}}$ are continuous$)\}$

$V_0 \longrightarrow V$

$$L(\underset{V_0}{f}) = f' \qquad \text{is it linear?}$$

yes $\qquad L(f + \alpha g) = (f + \alpha g)' = f' + \alpha g'$

Yes $L(f+\alpha g) = (f+\alpha g)' = f' + \alpha g'$

Δ $\downarrow$ Laplacian ∇ - - $\nabla\times$ are all linear

$V \to \mathbb{R}$ $V = L^1(\mathbb{R})$ $f \in V = L^1(\mathbb{R})$

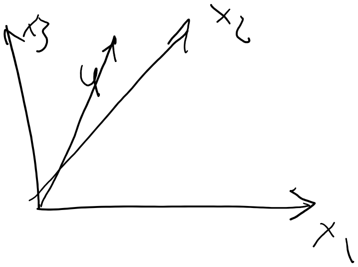
$$L(f) = \int_{-\infty}^{+\infty} f(x)\,dx \le \int_{-\infty}^{+\infty} |f(x)|\,dx < \infty$$

$$L(f+\alpha g) = \int (f+\alpha g)(x)\,dx = L(f) + \alpha L(g)$$

V inner product space, $\boxed{W = \mathbb{R}}$

we ~~only~~ only focus on $W = \mathbb{R}$

$V = \mathbb{R}^3$



$$u = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in V$$

$$\underset{\in R}{L_a(u)} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = a_1u_1 + a_2u_2 + a_3u_3 = a_i u_i = a\cdot u$$

for example $a = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$

$$L_a(u) = a\cdot u = 1u_1 + 2u_2 - 3u_3$$

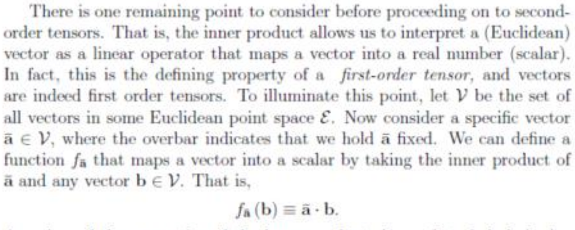
inner product space

$$\mathcal{V} \longrightarrow \mathbb{R}$$

linear function can be expressed as an inner product:

L is linear: $\exists\, a \ni L(u) = a \cdot u$ then we call L simply L_a

can find a



There is one remaining point to consider before proceeding on to second-order tensors. That is, the inner product allows us to interpret a (Euclidean) vector as a linear operator that maps a vector into a real number (scalar). In fact, this is the defining property of a *first-order tensor*, and vectors are indeed first order tensors. To illuminate this point, let $\mathcal{V}$ be the set of all vectors in some Euclidean point space $\mathcal{E}$. Now consider a specific vector $\bar{\mathbf{a}} \in \mathcal{V}$, where the overbar indicates that we hold $\bar{\mathbf{a}}$ fixed. We can define a function $f_{\bar{\mathbf{a}}}$ that maps a vector into a scalar by taking the inner product of $\bar{\mathbf{a}}$ and any vector $\mathbf{b} \in \mathcal{V}$. That is,

$$f_{\bar{\mathbf{a}}}(\mathbf{b}) \equiv \bar{\mathbf{a}} \cdot \mathbf{b}.$$

A review of the properties of the inner product shows that $f_{\bar{\mathbf{a}}}$ is indeed a linear operator. In fact, the *Riesz representation theorem* states that *every* linear function on $\mathcal{V}$ to $\Re$ can be represented in this fashion (by varying our choice of the fixed vector $\bar{\mathbf{a}}$)! We use a similar approach in the next section to define second-order tensors as a special class of linear operators.

Motivation σ^{ij} $F = \frac{\partial x}{\cancel{\partial X}}$ $F^{i}{}_{j}$

Background

$L : \mathcal{V} \longrightarrow \mathbb{R}$ linear functional

$\exists a$ $L(u) = a \cdot u$ $L_a = a \cdot$

$a = a^i e_i$ $v = v^i e_i$

vector expression

$$L_a(v) = (a^i e_i)(v^j e_j) = a^i v^j e_i \cdot e_j = \begin{cases} a^i v^i & \text{orthonormal basis} \\ a^i \underbrace{g_{ij}}_{\text{metric matrix}} v^j & \text{for general basis} \end{cases}$$

Here is where we introduce the concept of Dual Basis

$(\mathcal{V} \to \mathbb{R})$ it has a basis $e^1, e^2, \dots e^n$ where

$$e^i(e_j) = \delta^i_j \qquad \underset{\text{function from } V \to \mathbb{R}}{e^1}(\overset{\text{vector in } V}{e_1}) = 1 \quad e^1(e_2) = 0, \; \dots \; e^1(e_n) = 0$$

if we have a linear function L $V \to \mathbb{R}$

we can express it two different ways

$$1- \quad L(v) = \ell \cdot v = \boxed{\ell^i v^j g_{ij}} \qquad \ell = \ell^i e_i$$

or

$$2- \quad L(v) = \underbrace{(\ell_1 e^1 + \ell_2 e^2 + \dots)}_{\substack{\text{expand the function} \\ L \text{ in terms of the} \\ \text{basis of } V \to \mathbb{R}}}(v^1 e_1 + v^2 e_2 + \dots + v^n e_n) \qquad \ell = \ell_i e^i$$

$$= (\ell_i e^i)(v^j e_j) \underset{\text{because of linearity}}{=} \ell_i v^j \underbrace{e^i(e_j)} = \ell_i v^j \delta^i_j = \boxed{\ell_i v^j}$$

$$\sigma \qquad \tilde{\sigma}^{ij} e_i \otimes e_j \qquad \sigma_{ij} e^i \otimes e^j \qquad \sigma^i{}_j e_i \otimes e^j \qquad \sigma_i{}^j e^i \otimes e_j$$

For the rest of the course we won't distinguish between up or down indices, dual basis and space and actual vector space (and its basis)

For orthonormal basis, there is no fundamental difference whether to use up or down indices.

S, T Linear functions $V \to W$

$(S + T)$: $\ni \forall u \quad (S+T)(u) = Su + Tu$

$\underset{\in \mathbb{R}}{\lambda} S \quad \ni \forall u \quad (\lambda S) u = \lambda (S(u))$

$0(0_V) = 0_W$

We can show the space of linear functions is a vector space itself!

$S, T, U \in V \xrightarrow{\text{linear}} W \qquad \lambda, \mu \in \mathbb{R}$

A1) $S + T = T + S$

A2) $S + (T + U) = (S + T) + U$

A3) $S + 0 = S$

A1 - A3, B1 - B4 were properties of a vector

We can prove all of these for linear operators by going through super boring proofs that I'd skip

B1) $(\lambda\mu) S = \lambda(\mu S)$

B2) $\lambda(S + T) = \lambda S + \lambda T$

B3) $(\lambda + \mu) S = \lambda S + \mu S$

B4) $1 S = S$

$$\begin{tikzcd} \overset{V}{\mathbb{R}^3} \arrow[r, "\text{linear}"] \arrow[d] & \overset{W}{\mathbb{R}^3} \\ (e_1, e_2, e_3) & \end{tikzcd}$$

$$L = \begin{bmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{bmatrix} = l_{11}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + l_{12}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ldots$$

$$+ l_{33}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Motivation of how to define the basis

decent basis for $\mathbb{R}^3 \xrightarrow{\text{lin}} \mathbb{R}^3$ is $\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$

as we can see it's dimension is 9.

$U, V \in V \rightarrow$ inner product space

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \cancel{v} \end{bmatrix}$$

Inner product $U^T \cancel{V} = [u_1 \ \ u_2 \ \ u_3] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = u_i v_i = u \cdot v$ Inner product

Motivation

$$\underbrace{u \otimes v}_{\text{dyadic product}} = u v^T = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} u_1 v_1 & u_1 v_2 & u_1 v_3 \\ u_2 v_1 & u_2 v_2 & u_2 v_3 \\ u_3 v_1 & u_3 v_2 & u_3 v_3 \end{bmatrix}$$

very clear but uses components of u, v
X

Definition of dyadic product:
For vectors u, v in V, dyadic product is an operator from V -> V defined such that

$$\forall w \in V \quad (u \otimes v)\, w := \underline{(v \cdot w)\, u}$$

inner product

Actual definition

no component is used

good definition

Motivation

$$= \left(\underset{3\times 1}{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}} \underset{1\times 3}{\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}} \right) \underset{3\times 1}{\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}} =$$

$$\underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}}_{u} \Bigg(\underbrace{\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}}_{\substack{v \cdot w \\ \text{scalar}}} \Bigg)$$

Next time

would look like

$$e_1 \otimes e_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$T = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ & & \\ T_{31} & & T_{33} \end{bmatrix} = T_{11} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{e_1 \otimes e_1} + T_{12} \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{e_1 \otimes e_2} + \cdots + T_{33} \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{e_3 \otimes e_3}$$

$$T = T_{ij}\, e_i \otimes e_j$$

we'll prove it next time